Optimal Timing of Side Payments in the Infinitely Repeated Prisoner’s Dilemma

(This version serves as a web appendix to “On the Optimal Degree of Cooperation in the Repeated Prisoner’s Dilemma with Side Payments”)

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In studies of repeated games with transferable utilities, the timing of transfers is usually assumed to be exogenously set. In our formal analysis (Fong and Surti (2007); FS henceforth), we adopted a similar assumption; viz., that players make side payments right before the action stage in every period. Here we relax that assumption by allowing players to make side payments at any time of their choosing, taking the moment just preceding the first action stage as the beginning of time. More precisely, we think of a situation where there are finitely many points of time, \((n - 1)\), between every two consecutive action stages, and one additional point before the first action stage, at which players can make the side payment(s). The duration between any two points in time is therefore \(1/n\). We look at the case of \(n \rightarrow \infty\). We assume that the timing of side payments is selected and agreed upon before the game begins to achieve the most efficient outcomes. If players are free to choose when to make the side payments, they will make use of the moment right before the first period action stage to freely split the aggregate surplus of the whole game. Given this fact, we may assume that players maximize \(\frac{U_{I1}}{1-\delta_I} + \frac{U_{P1}}{1-\delta_P}\), subject to both players’ IC and IR constraints. As in FS, we will refer to SPE play paths that maximize this sum (across the set of SPE play paths) as Pareto dominant SPE.

In between the action stages of two consecutive periods, players have to decide whether to pay right after the action stage of the first period or wait until right before the action stage of the following period, or pick a point somewhere in between. The optimal time of the side payments depends on who the designated pay is. If the designated payer is the impatient player, it is most efficient to have him postpone the side payment to until the moment right before the action stage of the following period. In other words, if players agree to play partial cooperation and support it by the impatient player paying the patient player, side payment takes place right before every stage game, as is assumed in FS. If they are to achieve full cooperation and the patient player is the designated payer, then it is more efficient to have her pay immediately after each action stage. In other words, the side payment takes place
both right before and right after the first action stage under full cooperation. With these in mind, we proceed to characterize the Pareto dominant SPE payoffs for the whole unit square of discount factors and show that the results in FS remains valid qualitatively.

From the preceding discussion, we can see that $\overrightarrow{DC}$ is still a candidate Pareto dominant SPE play path. Moreover, when $\delta_I \geq \delta^*$, $\overrightarrow{CC}$ is also a candidate Pareto dominant play path. However, when $\delta_I < \delta^*$ and player $P$ must be the payer to support full cooperation in equilibrium for Pareto dominance; it is more efficient for the side payment to be delivered immediately after each action stage. Denote by $\overrightarrow{CC}$ a play path in which players play $(C,C)$ every period and $P$ pays $I$ immediately after each action stage.

Let $b$ be the stationary net side payment that is delivered after each action stage by $P$ to $I$ to support $\overrightarrow{CC}$, and that starts immediately after the first action stage. In order for this arrangement to be part of a SPE, let us begin by considering the incentive constraints at the action stage, respectively for players $I$ and $P$:

$$(1 - \delta_I)z + \delta_I x \leq (y + b), \text{ i.e., }$$

$$b \geq z - y - \delta_I (z - x),$$

$$(1 - \delta_P)z + \delta_P x \leq (y - b), \text{ i.e., }$$

$$b \leq \delta_P (z - x) - (z - y).$$

There exists $b \in \left[ z - y - \delta_I (z - x), \delta_P (z - x) - (z - y) \right]$ satisfying both players’ IC constraints if and only if

$$z - y - \delta_I (z - x) \leq \delta_P (z - x) - (z - y)$$

$$\frac{\delta_I + \delta_P}{2} \geq \frac{(z - y)}{(z - x)} = \delta^* \quad \text{(A1)}$$

The minimum side payment that supports incentive compatibility of both players is also the most efficient level due to the difference in discount factors:

$$b \equiv z - y - \delta_I (z - x) \begin{cases} > 0 \quad \text{if } \delta_I < \delta^*, \\ < 0 \quad \text{if } \delta_I > \delta^*. \end{cases}$$

Next, let us analyze the IC constraints at the side payment stage. When $\delta_I < \delta^*$, it is incentive
compatible for the patient player to pay at the side payment stage if and only if

\[
\delta_p x \leq (1 - \delta_p) \left( -\frac{b}{2} + \delta_p (y - b) \right)
\]

\[
\Leftrightarrow \delta_p \geq \frac{b}{y - x} = \frac{z - y - \delta_I (z - x)}{y - x}
\]

\[
\Leftrightarrow \delta_I \geq \delta^* - \delta_P \frac{y - x}{z - x}
\]

Suppose \(\delta_I > \delta^*\), it is always incentive compatible for the impatient player to pay at the side payment stage because

\[
(1 - \delta_I) \frac{b}{2} + \delta_I \left( y + b \right) = (1 - \delta_I) \left( z - y - \delta_I (z - x) \right) + \delta_I \left( y + z - y - \delta_I (z - x) \right) = (z - y) (1 - \delta_I) + \delta_I x > \delta_I x.
\]

Next, we show that whenever it is incentive compatible for players to play \((C, C)\), it is also incentive compatible for the patient player to pay. First, note that (A1) can be rewritten into \(\delta_I \geq 2\delta^* - \delta_P\).

Since

\[
2\delta^* - \delta_P - \left( \delta^* - \delta_P \frac{y - x}{z - x} \right)
\]

\[
= \frac{z - y}{z - x} - \delta_P \left( \frac{z - y}{z - x} \right) = \delta^* (1 - \delta_P) \geq 0,
\]

\(\delta_I \geq 2\delta^* - \delta_P\) implies \(\delta_I \geq \delta^* - \delta_P \frac{y - x}{z - x}\). Summing up, we have the following.

**Lemma A1** \(\overrightarrow{CC}\) is a SPE play path if and only if (i) \(\delta_I \geq \delta^*\), or (ii) \(\delta_I < \delta^*\) and \(\delta_I + \delta_P \geq \delta^*\).

Our approach is the following: we want to demonstrate that even if the choice of time to deliver the side payment lies with the players, for a nondegenerate subspace of discount factors, the choice will render the SPEs identified as Pareto dominant by FS will continue to remain so. In order for this to be the case, we have to characterize conditions under which either \(\overrightarrow{CC}\), \(\overrightarrow{DC}\) or \(\overrightarrow{CC}\) is Pareto dominant. The next result demonstrates that \(\overrightarrow{CC}\) can be supported as a SPE play path for a larger set \((\delta_I, \delta_P)\)-pairs than \(\overrightarrow{CC}\) does, albeit is Pareto dominated by the latter for all \(\delta_I \geq \delta^*\).

**Lemma A2** For \(\delta_I < \delta^*\), \(\overrightarrow{CC}\) is a SPE play path only if \(\overrightarrow{CC}\) is a SPE play path. When both \(\overrightarrow{CC}\) and \(\overrightarrow{CC}\) are subgame perfect, \(\overrightarrow{CC}\) Pareto dominates \(\overrightarrow{CC}\) if \(\delta_I > \delta^*\), and \(\overrightarrow{CC}\) Pareto dominates \(\overrightarrow{CC}\) if \(\delta_I < \delta^*\).
Proof. Suppose $\overrightarrow{CC}$ is not a SPE play path. That means $\delta_P < 2\delta^* - \delta_I$. It can be verified that $(\delta_P - \delta_I)^2 > 0$ implies $\frac{\delta_I + \delta_P}{2} > \frac{2\delta_I\delta_P}{\delta_I + \delta_P}$. Following that, we have

$$\frac{2\delta_I\delta_P}{\delta_I + \delta_P} < \frac{\delta_I + \delta_P}{2} < \delta^*$$

$$\Rightarrow \frac{1}{\delta_I} + \frac{1}{\delta_P} < \frac{2}{\delta^*}$$

and thus $\overrightarrow{CC}$ fails to be a SPE play path.

The last part is straightforward: $\overrightarrow{CC}$ and $\overleftarrow{CC}$ differ only in the timing of the side payments. When $\delta_I \geq \delta^*$, $I$ is the designated payment and it is more efficient to pay as in the case of $\overrightarrow{CC}$. When $\delta_I < \delta^*$, $P$ is the designated payer so it is more efficient to pay following the timing of $\overleftarrow{CC}$.

From Lemma A2, we see that while condition (A1) shares the same intuition of condition (3) that we derived in FS that players must be on average as patient as $\delta^*$ for full cooperation to be supported in equilibrium, we can see that equation (A1) is met by a larger set of $(\delta_I, \delta_P)$.

When $\delta_I \geq \delta^*$, we know that $\overrightarrow{CC}$ is Pareto dominated. So from FS, we know that the Pareto dominant play path is either $\overrightarrow{DC}$ or $\overleftarrow{CC}$. $\overleftarrow{CC}$ is Pareto dominant if $\frac{\delta_P}{\delta_I} < \frac{y}{z-y}$ and $\overrightarrow{DC}$ is Pareto dominant if $\frac{\delta_P}{\delta_I} > \frac{y}{z-y}$. If $x + y > z$, then $\overleftarrow{CC}$ is Pareto dominant for all $\delta_I \geq \delta^*$.

For the remainder of the analysis, we focus on $\delta_I < \delta^*$ where the Pareto dominant play path is either $\overrightarrow{DC}$ or $\overleftarrow{CC}$. The Pareto frontier of the set of SPE payoffs supported by $\overrightarrow{CC}$ is characterized by

$$\frac{U_{I1}}{1-\delta_I} + \frac{U_{P1}}{1-\delta_P} = \frac{z - \delta_I(z-x)}{1-\delta_I} + \frac{2y - (z - \delta_I(z-x))}{1-\delta_P}.$$  

Recall that $\overrightarrow{DC}$ can be supported as a SPE play path if and only if $\delta_P \geq x/(z-x)$ and the Pareto frontier achieved by $\overrightarrow{DC}$ is characterized by

$$\frac{U_{I1}}{1-\delta_I} + \frac{U_{P1}}{1-\delta_P} = \frac{x}{1-\delta_I} + \frac{z-x}{1-\delta_P}.$$  

We observe that $\overrightarrow{DC}$ Pareto dominates $\overleftarrow{CC}$ if and only if

$$\frac{z - \delta_I(z-x)}{1-\delta_I} + \frac{2y - (z - \delta_I(z-x))}{1-\delta_P} \leq \frac{x}{1-\delta_I} + \frac{z-x}{1-\delta_P} \Rightarrow \frac{\delta_I}{1-\delta_I} \leq \frac{2y - z}{z-x}.$$  

Notice that $\delta^* < \frac{x}{z-y}$ if and only if $x + y > z$ and recall that $\overleftarrow{CC}$ is Pareto dominant for all $\delta_I \geq \delta^*$ if and only if $x + y > z$. Now we are ready to present the full characterization of the Pareto dominant path for all discount factor pairs. With further algebraic manipulations, we can verify the following finding.
Proposition A1 Case (i) \( x + y < z \). \( \overrightarrow{DC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in \left[0, \max \left\{ \frac{z - y}{z}, \frac{\delta_P - 2y - z}{z - x}, \min \{2\delta^* - \delta_P, \delta_P\} \right\} \right] \times \left[\frac{x}{z - x}, 1\right];
\]

\( \overrightarrow{CC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in \left[\max \left\{ \frac{\delta_P - 2y - z}{z - x}, 2\delta^* - \delta_P \right\}, \delta^* \right] \times \left[\frac{y}{z - x}, 1\right];
\]

and \( \overrightarrow{CC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in \left[\max \left\{ \frac{z - y}{z}, \frac{\delta_P - 2y - z}{z - x}, \min \{2\delta^* - \delta_P, \delta_P\} \right\} \right] \times \left[\delta^*, 1\right].
\]

Case (ii) \( x + y > z \). \( \overrightarrow{DC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in \left[0, \max \left\{ \frac{\delta_P - 2y - z}{z - x}, 2\delta^* - \delta_P \right\} \right] \times \left[\frac{x}{z - x}, 1\right];
\]

\( \overrightarrow{CC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in \left[\max \left\{ \frac{\delta_P - 2y - z}{z - x}, 2\delta^* - \delta_P \right\}, \delta^* \right] \times \left[\delta^*, 1\right];
\]

and \( \overrightarrow{CC} \) is Pareto dominant if

\[
(\delta_I, \delta_P) \in [\delta^*, \delta_P] \times [\delta^*, 1].
\]

The following figures provide a visual characterization described in Proposition A1. Figure A1 depicts part (i) of the proposition and figure A2 depicts part (ii) of the proposition:

Figure A1: Characterization of Subgame Perfect Equilibria: \( x + y < z \)
Next, we point out both partial and full cooperation are nondegenerate. Because $\delta^* < 1$, from the above figures, we clearly see that $\overrightarrow{CC}$ is Pareto dominant for a non degenerate set of $(\delta_I, \delta_P)$-pairs. By studying both figures, we also see that $\overrightarrow{CC}$ is Pareto dominant for a non degenerate set of $(\delta_I, \delta_P)$-pairs if $\delta^* < \min \left\{ \frac{z-y}{z-x}, 1 \right\}$ which must hold because $2y > z$. Finally, $\overrightarrow{DC}$ is Pareto dominant for a non degenerate set of $(\delta_I, \delta_P)$-pairs because $\frac{x}{z-x} < 1$ and at $\delta_P = 1$,

$$\max \left\{ \min \left\{ \frac{z-y}{z} \delta_P, \delta_P - \frac{2y-z}{z-x} \right\}, \min \{2\delta^*-\delta_P, \delta_P\} \right\} = \min \left\{ \frac{z-y}{z}, 1 - \frac{2y-z}{z-x} \right\} = \frac{z-y}{z} > 0.$$

One can also verify that partial cooperation Pareto dominates full cooperation when both partial and full cooperation are subgame perfect equilibrium outcomes for a non degenerate set of $(\delta_I, \delta_P)$-pairs:

$$\begin{cases} \max \{0, 2\delta^* - \delta_P\}, \min \left\{ \frac{z-y}{z} \delta_P, \delta_P - \frac{2y-z}{z-x} \right\} \times \left[ \frac{z}{2(z-x)}, 1 \right] & \text{if } x + y < z, \\ \max \{0, 2\delta^* - \delta_P\}, \delta_P - \frac{2y-z}{z-x} \times \left[ \frac{z}{2(z-x)}, 1 \right] & \text{if } x + y \geq z. \end{cases}$$

References