Loyalty Rewards Facilitate Tacit Collusion

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Abstract

Using a dynamic overlapping-generations model, we show that loyalty rewards robustly facilitate tacit collusion. We compare the sustainability of tacit collusion when uniform prices are used, when loyal customers are rewarded without using commitment, and when loyalty rewards are implemented by committing to offering customers either lower fixed repeat-purchase prices or fixed repeat-purchase discounts. We find that, relative to uniform prices, rewarding loyalty without using commitment on the equilibrium path makes tacit collusion easier to sustain, because a deviating firm is unable to steal one period of industry profit before losing all future profits. When loyalty rewards are offered by firms committing to repeat-purchase prices, collusion is even easier to sustain, since a deviating firm cannot renege on its discounted price for repeat-purchase customers. When firms commit to repeat-purchase discounts, they also commit to lowering the price for their repeat-purchase customers if they undercut the regular price, rendering tacit collusion to be even more readily sustainable. Our results hold whether products are homogeneous or horizontally differentiated as in a Hotelling model.

Keywords: Loyalty Rewards, Repeat-purchase Prices, Repeat-purchase Discounts, Tacit Collusion

JEL Codes: D43, L13

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1 Introduction

Loyalty programs are prevalent. A well-known example is the frequent-flier programs offered by airline companies. All major U.S. airlines have frequent flier programs (FFPs). The largest programs, at American, United and Delta Airlines, have more than 20 million members each, and on average about 5% of an airline’s seats are allocated to FFP members using award tickets (http://frequentflier.com). Loyalty programs can take various forms. The most popular method of rewarding loyal customers involves frequent-shopper programs, where consumers receive certain rewards/discounts (e.g., a free flight or hotel stay) after reaching certain purchase thresholds.

One important observation is that most loyalty programs can be viewed as variants of repeat-purchase discounts, where customers are entitled to purchase another unit of a (possibly different) product at a (sometimes 100%) discount after an initial purchase or multiple purchases. As a result, loyalty programs and repeat-purchase discounts are often viewed as qualitatively similar. This paper will also take this stance, and we consider the retailers’ practice of sending coupons to customers in their database, i.e., prior customers, to induce repeat-purchase to be a type of loyalty program (e.g., the apparel retailer New York & Company). Another example involves promotions included inside product boxes for future-purchase use (e.g., Huggies and Pampers).

There are several theoretical studies in the literature that analyze loyalty programs or repeat-purchase discounts. Most existing work is based on two-period models, and the results of these studies indicate that the impact of such discounts on competition is mixed and sensitive to changes in assumptions. While Banerjee and Summers (1987) and Kim, Shi and Srinivasan (2001) find that such programs/discounts are anti-competitive, Caminal and Matutes (1990) and Caminal and Claici (2007) argue that these programs/discounts tend to increase business stealing and, thus, promote competition. The difference in these findings can be attributed to the difference in the assumptions made regarding consumer heterogeneity, product heterogeneity, the change in consumer taste from the first to the second period, the types of discounts/coupons being offered, and the market structure (number of firms).

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1 Other examples include frequent-guest programs at hotels, cash back for credit card purchases, and similar programs for patronage at restaurants, coffee shops, salons etc. Frequent-shopper programs for grocery retailing are often different from other loyalty programs since a grocery shopper can, without previous-purchase obligation, enjoy all the benefits of membership as long as he/she has a membership card, applies for one, or even just borrows the cashier’s card at the time of purchase. Strictly speaking, grocery shopper programs are not loyalty programs because benefits are not limited to loyal customers.

2 For example, see Banerjee and Summers (1987), Kim, Shi and Srinivasan (2001), Caminal and Matutes (1990), and Caminal and Claici (2007).
It is surprising that while in some studies it has been argued that loyalty programs may facilitate tacit collusion, no study of loyalty programs is based on a fully dynamic model, which is the workhorse of the traditional analyses of tacit collusion. The main purpose of this paper is to show that in a fully dynamic framework, it is a robust insight that loyalty rewards facilitate tacit collusion. We analyze a market in which infinitely-lived firms serve a constant flow of finitely-lived customers who arrive in overlapping generations. We compare subgame perfect Nash equilibria (SPNE) in which firms price uniformly on the equilibrium path and SPNE in which firms reward loyal (repeat-purchase) customers on the equilibrium path. We prove that firms earn supranormal profits for a wider range of discount factors (for any number of firms) and a larger number of firms (for any discount factor), when they offer loyalty rewards on the equilibrium path. In particular, the use of loyalty rewards enhances firm profits and lower consumer surplus by facilitating tacit collusion even when both the consumers and products are homogeneous. In contrast to existing findings established in two-period models, our qualitative results, which indicate that loyalty rewards facilitate tacit collusion, are independent of whether firms can commit to rewarding loyal customers, independent of the type of rewards to which firms commit (the future price or the future discount relative to the regular price), and independent of the market structure. These factors only affect the extent to which loyalty rewards facilitate collusion. We compare various classes of subgame perfect equilibria to illustrate how the use of repeat purchase discounts generally softens competition by making tacit collusion sustainable for a wider

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3 A section in Caminal and Claici (2007) analyzes an overlapping-generations model where firms *once-and-for-all* simultaneously commit in the first period to prices in all periods. Although the time horizon is infinite, since firms only act in the first period, this game form does not allow for tacit collusion. Liu and Serfes (2007) study tacit collusion in an infinitely repeated Hotelling duopoly game. In each period, firms have access to customer information, which allows them to segment the consumers into various groups. Firms can discriminate among these groups in each period, but unlike those in the current paper, do not intertemporally price discriminate based on purchase history. They find that collusion is more difficult to sustain when the quality of customer information increases, i.e., when customers are segmented into more groups.

4 Although Banerjee and Summers also find that intertemporal price discrimination through couponing can be anti-competitive in a two-period model when consumers are homogeneous, their finding relies on the special assumption that firms set prices sequentially within each period according to an exogenous order. In the working paper version of this article, we show that in a two-period model similar to theirs except that firms set prices simultaneously in each period, couponing is not anti-competitive.

5 We show that offering repeat-purchase discounts without using a commitment device and offering such discounts by providing new customers with either repeat-purchase prices or repeat-purchase discounts all make collusion easier to sustain, relative to uniform pricing. Moreover, unlike the literature where loyalty rewards typically reduce competition only when consumers are heterogeneous, we derive our results in a model of homogeneous consumers.
range of discount factors and among a larger number of firms.

When firms price uniformly across generations in an overlapping-generations setting, their ability to tacitly collude is no different from those who sell to short-lived consumers. It is a simple but useful observation that in both cases, a deviating firm can capture one period of industry profit before losing all future profits. Because firms are making the same tradeoff in both cases, the conditions for the sustainability of profitable tacit collusion are the same.

Now suppose that firms raise the price for new customers and lower the price for old customers so that each customer’s total discounted price over her lifetime, and thus, firm profits, remains unchanged. Consumers pay a higher (regular) price for an initial purchase but receive a discount for a subsequent purchase. Note that loyal customers are rewarded (relative to new customers) because, for their repeat-purchases, they pay less than new customers. Also suppose such rewards are offered without any commitment by the firms; these firms can sell to customers at a high price in one period and then eliminate the discount these customers expect to receive in the following period. When a firm deviates, it can undercut the regular price by an infinitesimal amount to steal its competitors’ profits from new customers and, due to the lack of commitment, it can simultaneously renege on its discount for repeat-purchase customers to increase its profit further. Nevertheless, as long as the price for its repeat-purchase customers is not substantially lower than the regular price, so that the deviating firm cannot benefit too much from eliminating the discount for repeat-purchase customers, the fact that the deviating firm does not capture its competitors’ repeat-purchase customers through such a deviation prevents it from capturing a profit as large as one period of the industry profit before losing its profits in all future periods. Alternatively, the deviating firm can cut its price further to attract all customers, including its competitors’ old customers. Although everyone who has an instantaneous demand will buy from the deviator, new customers are paying the deviator the same discounted price as repeat-purchase customers do. As a result, the deviating firm is still unable to steal one period of the industry profit before losing all of its future profits. Therefore, deviation becomes less attractive when firms offer appropriately chosen repeat-purchase discounts and tacit collusion can be sustained for a wider range of discount factors.

While rewarding loyal customers without commitment already facilitates tacit collusion, if firms implement loyalty rewards by committing to a (lower) fixed repeat-purchase price or a fixed repeat-purchase discount off the regular price (e.g., by issuing coupons to the customers at the time of the initial purchase), the deviating firm is prevented from eliminating the discount to its repeat-purchase customers during the period when it chooses to undercut the regular price. The commitment not to renege on the discounted price for repeat-purchase customers further lowers the deviation payoff in comparison to the case when firms reward
loyalty without using commitment. This is clearly the case when firms commit to their new customers that they will charge them a fixed repeat-purchase price which is lower than the future regular price. When firms instead commit to their new customers a fixed amount off the future regular price, they also commit to lowering the price for their repeat-purchase customers if they cut the regular price. This further lowers the deviation profit and renders tacit collusion to be sustainable for an even larger range of discount factors.

There are some fundamental differences between our analysis of loyalty rewards in a fully dynamic model and existing analyses based on two-period models both in terms of approaches and findings. In a two-period model, old customers in the second period are entitled to repeat-purchase discounts. Although repeat-purchase discounts lower firms’ deviation payoffs in the second period just like exogenous switching costs do, offering these repeat-purchase discounts is costly for the firms. After netting repeat-purchase discounts or the costs of offering loyalty benefits, firm profits may go up or down, depending on the specification of the model. This explains why the effect of repeat-purchase discounts on firm profits in a two-period model is sensitive to changes in assumptions.\(^6\)

In contrast, we analyze a fully dynamic model illustrating how implementation of loyalty programs impacts the effectiveness of non-Markov strategies, which cannot be analyzed in a two-period model. We find that for any given profit level (including the monopoly profit), tacit collusion is sustainable (by non-Markov strategies) for a wider range of discount factors and among a larger number of firms when firms offer loyalty rewards on the equilibrium path. This result is robust in the sense that it holds regardless of the specific type of loyalty rewards being offered and the market structure. Loyalty programs allocate life-cycle profits from each generation of customers unequally over the two periods of their lifetime in the market, with firms charging customers a higher price initially and a lower price for a subsequent purchase. We believe it is a general property that because of this unequal allocation of profits, when a firm deviates, it is forced to choose between two options: stealing only the business of new customers at higher price or substantially lowering the regular price to also steal its competitors’ established customers. Either way, the deviator is not able to capture the entire industry profit for one period before losing all future profits as it can do in the absence of loyalty programs. This reduces firms’ incentive to deviate, and thus facilitates tacit collusion. We discuss in Section 4 and prove formally in the Appendix that this insight generalizes to a model of differentiated products where consumers have heterogeneous firm

\(^6\)We implicitly assume that consumers have heterogeneous firm preferences when referring to two-period models here. In many cases loyalty discounts play no role in a two-period model with homogenous products, since firms always earn zero profits in both periods regardless of whether they are allowed to offer loyalty discounts or not.
preferences as in a Hotelling model.

The rest of the paper is organized as follows. We review the related literature in Section 2. Section 3 provides the model and Section 4 contains the analysis for uniform pricing and various types of loyalty programs. In Section 5, we discuss several extensions and we conclude in Section 6. The Appendix contains a detailed analysis for the case of differentiated products.

2 Related Literature

There is a growing literature on loyalty programs. One strand of this literature has illustrated that loyalty programs reduce competition and improve profits. For example, Banerjee and Summers (1987) consider a two-period duopoly model with homogeneous consumers. All consumers and firms live for two periods. In each period, firms set prices sequentially, with one of the firms designated to be the first mover. Banerjee and Summers find that loyalty programs help firms sustain the fully collusive outcome, where both firms charge the monopoly price in both periods. However, their analysis crucially relies on the assumptions that firms sequentially set prices in each period and no new customer enters the market in the second period. Furthermore, the anticompetitive effect of loyalty programs is found to be sensitive to the type of commitment.7

Kim, Shi, and Srinivasan (2001) also investigate a two-period model but with heterogeneous consumers who differ in firm preferences and length of lifetime in the market. In particular, heavy users live for two periods and have a different price sensitivity than light users, who only live for one period. The light users from the first period are replaced by a new cohort of light users in the second period. While the authors find that reward programs generally raise prices in both periods, after netting the costs of the rewards, firm profits can either increase or decrease depending on the relative fractions and price sensitivities of light users and heavy users.8

7Caminal and Matutes (1990, p.370) point out that if loyalty programs promised fixed prices instead of fixed discounts, then these programs would have no effect on firm profits in the setup of Banerjee and Summers (1987).

8Although Kim et al. (2001) interpret the anticompetitive effect of loyalty discounts as facilitation of tacit collusion, firms do not use non-Markov strategies in their analysis. A firm’s deviation in the first period affects the ensuing competition only by changing the distribution of repeat-purchase customers and the loyalty discounts. In contrast, in our analysis, tacit collusion refers to the firms’ coordination when using non-Markov strategies to achieve profits above the static Nash equilibrium profits and facilitation of tacit collusion means supporting the same industry profit for a larger range of discount factors. A firm’s deviation not only changes the distribution of repeat-purchase customers and loyalty discounts, but also triggers all
In Lal and Bell (2003), consumers purchase two products at the same time, but they only qualify for the loyalty reward if they purchase both products from a single firm.\(^9\) They find that this kind of loyalty program improves profits.

On the other hand, several studies have shown that loyalty programs may enhance competition. Caminal and Matutes (1990) consider a two-period duopoly model with heterogeneous consumers. The consumers’ preferences across the two periods are independent. The authors find that if firms precommit to a second-period price for their repeat-purchase customers, then equilibrium profits fall relative to the profits in the absence of a commitment. However, if they precommit to discounts for repeat-purchase customers, then the equilibrium profits increase. Caminal and Claiici (2007) extend the duopoly model to include \(n\) firms (where \(n\) is sufficiently large), using the spokes model developed by Chen and Riordan (2007). They find that loyalty programs soften competition only when firms precommit to second-period discounts and the number of firms is sufficiently small. Otherwise, loyalty programs necessarily intensify competition and lower profits, relative to the case where such commitment is not possible.\(^{10}\) Greenlee, Reitman, and Sibley (2007) consider a multi-product firm that is a monopolist in one market but faces competition in another market. They analyze the monopolist’s tactic of using a bundled loyalty discount program, one that requires its customers to meet purchase thresholds in multiple markets before qualifying for loyalty discounts, to increase its market share in the competitive market. They find that such bundled loyalty discounts in general have ambiguous effects on consumer surplus and total welfare.

Basso, Clements, and Ross (2009) analyze a duopoly model of frequent-flyer programs (FFPs) with moral hazard, which stems from the agency relationship between employers and employees in which the employers pay for the travel and the employees, who book the travel, enjoy the direct benefits of FFPs. FFPs induce travelers to buy expensive tickets from airlines offering generous FFP benefits. They find that a single firm adopting an FFP enjoys a large advantage. But when both firms adopt FFPs, competition through FFP benefits may be so intense that firm profits go down, despite the higher equilibrium prices.

By instead analyzing a fully dynamic framework, this paper shows that the qualitative anticompetitive effect of loyalty rewards is no longer sensitive to the way discounts are offered, the way consumer heterogeneity is modeled, and the number of firms involved. Moreover, firms to abandon both collusion and loyalty programs following the deviation.

\(^9\)There are cases where a consumer has to concentrate her purchases from a single seller to accumulate enough points for redemption before those points expire.

\(^{10}\)In all these studies, except Banerjee and Summers (1987), profits are functions of the unit transportation cost, which is a measure of product differentiation/consumer heterogeneity. When products/consumers are homogeneous, profits should go down to zero with or without loyalty programs, and loyalty programs should have no impact on profits.
a commitment to discounted repeat-purchase prices and a commitment to repeat-purchase discounts both soften competition, contrasting the findings by Caminal and Claici (2007). In fact, we find that even without commitment, the practice of giving discount to repeat-purchase customers still softens competition. Such a result is unlikely to be obtained in a two-period model.

Cairns and Galbraith (1990) also study an anti-competitive effect of loyalty rewards. In their study, when the incumbent firm’s loyalty program is valued higher to consumers than the potential entrant’s loyalty program (e.g., due to a larger network in the case of an airline company), it is difficult for the entrant to enter the market and steal business from the incumbent. This is somewhat similar to our paper, where loyalty programs make it more difficult for a deviating firm (instead of an entrant) to steal business. However, our analysis is different in many important ways. For instance, our analysis does not rely on asymmetry in firms’ abilities to create valuable loyalty rewards, and all firms (as opposed to only the firm with a superior loyalty program) benefit from the use of loyalty rewards. A subtler difference is that in our analysis, the loyalty reward does not reduce competition by reducing the number of firms, instead it allows the same number of firms to charge a higher price.

While the theoretical literature on loyalty programs has focused on loyalty programs’ impact on competitiveness of the market, the vast majority of empirical research on loyalty programs studies the performances of individual programs and the determinants of their performances (see e.g., Bolton, Kannan and Bramlett 2000, Lewis 2004, and Lederman 2007). As a rare exception, Liu and Yang (2009) analyze the interaction between market structure and the benefits of competing loyalty programs based on panel data in the airline industry. They find that as a market becomes less concentrated, firms are more likely to offer loyalty programs. They also show that when a product category has close substitutes from outside the industry, loyalty programs could help the industry gain competitive advantage over these substitutes. Nevertheless, direct evidence of the pro- or anti-competitive effects of loyalty programs is still lacking.

The literature has also studied the practice of charging a competitor’s loyal customers a lower price, which constitutes a form of intertemporal price discrimination opposite to repeat-purchase discounts. This is often called “paying customers to switch” or “customer poaching.” The rationale behind this pricing strategy is that a firm needs to provide discounts to its rival’s customers to attract them, either because these customers have a weaker preference for the firm’s product due to exogenous reasons or because of brand-switching costs. The studies by Chen (1997), Fudenberg and Tirole (2000), Taylor (2003), and Villas-Boas (1999) share the feature that while the unilateral use of customer poaching enhances a firm’s individual profit, in equilibrium, the industry profit, and thus the firm profits are low-
ered by such pricing practices. Since we assume consumers are semianonymous, firms cannot target competitors’ old customers to poach them. Also, in our main model consumers are homogenous, so there is no role for customer poaching.

The earlier literature on “customer poaching” studies static models where firms start with an exogenous partition of consumers, enabling them to segment consumers into identifiable groups and then price discriminate across these groups. Shaffer and Zhang (1995) and Bester and Petrakis (1996) analyze a duopoly model where each firm sends out coupons to consumers who prefer its rival firm’s product. It is shown that allowing firms to issue poaching coupons leads to a prisoners’ dilemma game, i.e., both firms issue poaching coupons in equilibrium and firm profits are lowered due to intensified competition. Shaffer and Zhang (2000), on the other hand, show that when demand is asymmetric, it may be optimal for one firm to reward its own customers, and such price discrimination may actually reduce competition.

3 The Model

There are \( n \) (\( n \geq 2 \)) infinitely-lived firms that sell homogeneous perishable products. Each firm has a marginal cost of zero.

Consumers arrive in overlapping generations. In each period, a continuum of consumers of measure one enter the market and each of them stays in the market for two periods. Each consumer demands up to one unit of the good in each period. A customer in the first period of her market life is called a new customer and a customer in the second period of her market life is called an old customer. Each consumer derives an instantaneous utility \( U \) from one unit of the good in each period of her life and derives zero instantaneous utility if she does not consume.\(^{11}\) We assume that firms can directly identify consumers who purchased from them in the previous period but cannot distinguish between new customers and competitors’ old customers.\(^{12}\) In other words, consumers are *semianonymous*, according to the definition in Fudenberg and Tirole (1998). When a consumer purchases from firm \( i \) in both periods of

\(^{11}\)When there is a substitute outside good which provides consumers a positive net surplus, then \( U \) should be interpreted as the incremental instantaneous utility above and beyond the utility from consuming the outside good. In that case, the actual instantaneous utility will be higher than \( U \).

\(^{12}\)This assumption is used only in Subsection 4.2 when we analyze firms’ use of intertemporal price discrimination without commitment. When firms charge uniform prices, they do not need to identify customers. When firms intertemporally price discriminate with commitment (e.g., by using coupons), again they do not need to identify customers, since repeat-purchase customers will identify themselves by presenting coupons. However, when firms intertemporally price discriminate without commitment, they need to be able to identify their own repeat-purchase customers.
her market life, we call this consumer firm $i$’s loyal customer or repeat-purchase customer.

Firms and consumers have a common discount factor, $\delta \in (0, 1)$.

We compare the sustainability of tacit collusion across cases in which firms employ different pricing strategies on the equilibrium path: uniform pricing, loyalty rewards without using commitment, and loyalty rewards with commitment to repeat-purchase prices or repeat-purchase discounts. When firms deviate, or more generally off the equilibrium path, firms can use any pricing strategy.\(^{13}\)

**Uniform Pricing** When firms use uniform pricing, in each period $t$, firms simultaneously set market prices $p_{i,t}$, $i = 1, 2, ..., n$.

**Rewarding Loyalty Without Using Commitment** When firms reward loyalty without commitment (e.g., without using coupons), firms simultaneously set market prices $\{p^1_{i,t}, p^2_{i,t}\}$ in each period $t \geq 2$, where $p^1_{i,t}$ is its regular price that everyone is entitled to, $p^2_{i,t}$ is firm $i$’s discounted price for repeat-purchase customers only, and $p^2_{i,t} < p^1_{i,t}$.

The above-mentioned pricing scheme is a form of intertemporal price discrimination favoring loyal customers. Note that firms do not commit to a future repeat-purchase discount when they sell to new customers. Each firm can cancel the discount to its repeat-purchase customers and switch to uniform pricing in any period.

Alternatively, firms can use a commitment to implement such loyalty rewards. They can promise loyalty rewards to their new customers by committing to a (lower) fixed price on a future purchase, or committing to a fixed discount off the regular price on a future purchase. Note that we consider a commitment that is conducted period by period to each new generation of customers instead of a once-and-for-all commitment to all future generations. In other words, in any period, a firm can stop making commitments to incoming or future customers but has to honor the rewards to its loyal customers whom it committed to in the previous period.

**Loyalty Rewards with Commitment to Fixed Repeat-Purchase Prices** When firms commit to a repeat-purchase price, in each period $t \geq 2$, each firm simultaneously sets a regular price, $p^1_{i,t}$, and promises its returning customers a fixed price, $p^2_{i,t+1}$, when they purchase in period $t + 1$.

**Loyalty Rewards with Commitment to Fixed Repeat-Purchase Discounts** When firms commit to a repeat-purchase discount, in each period $t \geq 2$, each firm simultaneously

\(^{13}\)An alternative approach of analysis is to compare the sustainability of tacit collusion in different settings: one in which only uniform pricing is permitted and others in which different loyalty rewards are permitted. In such analysis, it can be similarly shown that by allowing firms to reward loyalty, tacit collusion can be sustained for a wider range of discount factors.
sets a regular price, \( p_{i,t}^{1} \), and issues to its customers dollars-off coupons, which promise a discount of \( D_{i,t+1} \) dollars off the regular price in the following period. In other words, if in period \( t + 1 \) the firm charges a regular price of \( p_{i,t+1}^{1} \), then repeat-purchase customers pay \( p_{i,t+1}^{2} = p_{i,t+1}^{1} - D_{i,t+1} \).

We do not formally analyze repeat-purchase discounts in the form of percentage-off coupons. If a firm issued \( D\%-\)off coupons to its customers in period \( t \), the commitment for period \( t + 1 \) would be \( p_{i,t+1}^{2} = p_{i,t+1}^{1} (1 - D\%) \). Percentage-off coupons are qualitatively similar to dollars-off coupons. The analysis in Subsection 3.4 can easily be adapted to this case, however, this adaptation, is left to the interested reader.

Note that an implication of our assumption that consumers are semianonymous is that firms are unable to offer discounts targeted at consumers who previously purchased from a competitor. In practice, firms keep a record of their customers’ previous transactions and are able to offer exclusive discounts or benefits to them accordingly. When we apply our analysis to repeat-purchase discounts by the use of databases, our assumption essentially means that firms do not have access to their competitors’ databases, which they would need to match their competitors’ repeat-purchase discounts. When we apply our analysis to situations in which repeat-purchase discounts are offered through coupons, one interpretation of our assumption is that these coupons are relatively easy to forge and only the firm issuing the coupons can readily authenticate them.\(^{14}\)

In our analysis, both firms and consumers are forward looking: firms maximize lifetime discounted profits and consumers maximize lifetime discounted utilities. However, we allow them to differ in their levels of sophistication. There is common knowledge of rationality among firms so they correctly anticipate competitors’ pricing strategies in equilibrium. On the other hand, in order to capture the idea that consumers are not strategic players, we assume that they are not sophisticated enough to understand the strategies that firms use to support the equilibrium prices. More specifically, we assume that when consumers observe any deviation from the equilibrium prices, they believe that in the following period, the prices will return to the equilibrium level. As we will see in the tacitly collusive equilibria that we study, such belief simply implies that regardless of \( p_{1} \), by undercutting it by an infinitesimal amount, a deviating firm can attract all the new consumers. In Section 4, we will discuss

\(^{14}\)For example, shoppers may only be required to provide a coupon code to obtain a discount at a firm. While this firm can easily validate its own coupon codes, it cannot easily verify the authenticity of another firm’s coupon codes. Thus, it is easy for customers to forge other firms’ coupon codes when shopping at a firm that matches other firms’ coupons. As reported by ABC’s Good Morning America on July 9, 2008, “Fear of counterfeits sometimes causes stores to reject Internet coupons altogether.” (Source: http://abcnews.go.com/GMA/Parenting/story?id=5335397&page=1)
the alternative assumption that consumers correctly anticipate firms’ equilibrium strategies
and show that under the alternative assumption, our findings will be qualitatively the same
but quantitatively strengthened.

The remaining assumptions of our model are as follows. We assume that firms cannot
make their transaction prices/discounts contingent on each other’s posted prices/discounts.\textsuperscript{15}
We also impose the restriction that $p_1, p_2 \geq 0$, both on and off equilibrium paths. This restric-
tion implies that if $p_{1,t+1}^1 < D_{i,t+1}$ (so that $p_{1,t+1}^1 - D_{i,t+1} < 0$), then $p_{2,t+1}^2 = \max\{p_{1,t+1}^1 - D_{i,t+1}, 0\} = 0$. At the end of Subsection 4.4, we will discuss how this assumption impacts our analysis.
Finally, we assume that there is a public randomization device, so that when the industry
can sustain a profit level $\Pi$, the entire set of profits $[0, \Pi]$ is also attainable by applying
public randomization at the beginning of the first period. This assumption simplifies and
shortens some of our proofs but does not impact our findings qualitatively.

4 Analysis

In this section, for each feasible profit level, we compare the sets of discount factors for which
tacit collusion is sustainable when firms use uniform pricing on the equilibrium path and
when firms offer loyalty rewards with or without commitment on the equilibrium path. We
restrict our attention to stationary equilibrium paths.

Although firms behave differently on the equilibrium paths in these equilibria, in all
classes of equilibria, off the equilibrium path, firms have access to the same set of pricing
strategies: they can price uniformly; they can charge their repeat-purchase customers and
other consumers differently; or they can choose whether or not to commit to offering loyalty
rewards to an incoming new generation of customers. For ease of exposition, however, we
assume that if a firm deviates from any collusive price, starting in the following period, all
firms permanently revert to the static Nash equilibrium prices of $p_1^i = p_2^i = 0$ for all $i$. In
other words, the deviating firm receives zero profit on the punishment path.\textsuperscript{16}

\textsuperscript{15}If firms match each other’s prices and consumers can request price-matching without a hassle cost, then
the monopoly profit can be sustained even in a static game.

\textsuperscript{16}In the case when there are only two firms, if the deviating firm distributes $D$ dollars-off coupons at the
time of deviation, there is no pure strategy equilibrium in the ensuing punishment period. However, there is
still a mixed strategy equilibrium in which the deviating firm earns zero profit (while the non-deviating firm
earns a positive profit). In the mixed strategy equilibrium, firm 1 (the deviating firm) and firm 2 randomize
4.1 Tacit Collusion Supported by Uniform Pricing

Suppose that all firms charge a uniform price \( p \in [0, U] \) to both generations of consumers. Take any period \( t \geq 2 \). Since in each period there is a measure two of consumers, each firm’s discounted equilibrium profit is \( \frac{2p}{n(1-\delta)} \). By undercutting the equilibrium price, a deviating firm can earn an instantaneous profit arbitrarily close to \( 2p \) and a profit of zero thereafter. Therefore, firms have no incentive to deviate in period 2 and thereafter if and only if

\[
\frac{2p}{n(1-\delta)} \geq 2p \\
\delta \geq \delta^U \equiv 1 - \frac{1}{n}. \tag{1}
\]

Since the market is only half its size in period 1 (\( t = 1 \)), the incentive to charge the equilibrium price is obviously satisfied when (2) holds. Each firm’s instantaneous profit is \( p/n \) in the first period, so its payoff from the entire game is \( \frac{p}{n} + \frac{2\delta p}{n(1-\delta)} = \frac{1+\delta}{1-\delta} \frac{p}{n} \). Since \( p \in [0, U] \), the range of possible industry profits is

\[
\left[ 0, \frac{1+\delta}{1-\delta} U \right].
\]

Another way to understand the upper bound of the industry profit is that in each period a new generation of consumers arrive and firms can earn up to \( (1+\delta)U \) from each generation of consumers, giving rise to a present discounted value of \( (1+\delta)U / (1-\delta) \). We summarize our analysis as follows:

**Lemma 1** If \( \delta \in [\delta^U, 1) \), then any industry profit \( \Pi \in [0, \frac{1+\delta}{1-\delta} U] \) is sustainable as a SPNE in which firms price uniformly. If \( \delta \in (0, \delta^U) \), then there is no profitable SPNE in which firms price uniformly.

It is clear that the condition for the sustainability of collusion with uniform pricing is identical to the condition when consumers are short-lived. This is because in both cases a deviating firm can capture one period of the industry profit before losing all future profits.

\( p_1 \) and \( p_2 \) on the support \([D_2, D]\) according to the following CDFs:

\[
F_2(p) = \begin{cases} 
\frac{2p-D}{p} , & p \in \left[ \frac{D}{2} , D \right], \\
1 - \frac{D}{p} , & p \in \left[ \frac{D}{2} , D \right), \\
1 , & p = D.
\end{cases}
\]

As a result, issuing coupons during deviation does not increase the deviation profit. Since our focus is on on-the-equilibrium path behavior and equilibrium payoffs, assuming reversion to marginal cost pricing is without loss of generality.
On the other hand, this observation also points to the fact that if firms can use other pricing strategies to prevent a deviating firm from capturing one period of the industry profit before losing its share of future profits, then tacit collusion will become easier to sustain.

4.2 Tacit Collusion Supported by Loyalty Rewards Without Commitment

Recall our assumption that consumers are semianonymous so that firms are able to offer exclusive discounts to their repeat-purchase customers but cannot further price discriminate. To be more concrete, imagine that when a consumer buys from a firm, the firm enters the consumer’s information into its database. This database allows the firm to identify the same consumer as a loyal customer in the next period. However, it cannot distinguish between new customers and its competitors’ old customers, since neither has been entered into its database.

Suppose that in each period, all firms charge a pair of prices \((p_1, p_2)\). For each firm, \(p_2\) is the price for its loyal customers. All other customers, including new customers and competitors’ old customers, pay \(p_1\). We focus on equilibria in which \(p_2 < p_1\). If \(p_2 > p_1\), then all old customers will either switch firms or not buy at all, making \(p_2\) irrelevant, and \(p_2 = p_1\) is uniform pricing. In any period \(t \geq 2\), each firm can earn an equilibrium profit of \(p_2/n\) from old customers and \(\pi/n(1-\delta) = \frac{p_1 + \delta p_2}{n(1-\delta)}\) from new and future customers, where \(\pi = p_1 + \delta p_2\) is the industry profit from each generation of customers. Each firm’s collusive profit is

\[
\frac{p_2}{n} + \frac{p_1 + \delta p_2}{n(1-\delta)} = \frac{p_1 + p_2}{n(1-\delta)}.
\]

For consumers to be willing to purchase in the first period of their market life, it is necessary that \(p_1 + \delta p_2 \leq (1 + \delta) U\). Note that \(p_2 \leq p_1\) and \(p_1 + \delta p_2 \leq (1 + \delta) U\) together imply that \(p_2 \leq U\). Also since \(p_2 \geq 0\), the upper bound on \(p_1\) is \((1 + \delta) U\).

Figure 1 depicts how firms, starting in period 2, simultaneously serve two “submarkets,” one for new customers and one for old customers.

In this subsection, we focus on the special case where loyalty rewards are offered without using commitment. In other words, at the time when a new consumer pays \(p_1\) in the first period of her market life, there is no guarantee that the firm will charge her \(p_2 < p_1\) in her repeat purchase in the following period. Although in real life firms do commit to the loyalty reward a consumer receives at the time of the initial purchase, it is useful to study the special case here before analyzing loyalty rewards with commitments. Doing so enables us to demonstrate the sheer fact that firms intertemporally price discriminate over a consumer’s
lifetime already facilitates tacit collusion and how the use of different commitments further enhances the sustainability of tacit collusion.

In the previous subsection, we showed that when firms uniformly set \( p_1 = p_2 = p \), any industry profit \( \Pi \in [0, \frac{1+\delta}{1-\delta}U] \) is sustainable if and only if \( \delta \geq 1 - \frac{1}{n} \). In this subsection, we will demonstrate that there exist subgame perfect Nash equilibria (SPNE) with \( p_2 < p_1 \) which support the same range of industry profit for a wider range of discount factors. When \( p_2 < p_1 \), a deviating firm can either undercut \( p_1 \) to steal only new customers or undercut \( p_2 \) to steal all customers, including its competitors’ old customers. In fact, when it only undercuts \( p_1 \), it can also renge on the discount it offered to repeat-purchase customers in the past. However, we will show that if \( p_1 \) and \( p_2 \) are appropriately chosen, then the deviating firm always fails to steal one period of industry profit before losing all future profits. Therefore, tacit collusion is sustainable for a wider range of discount factors.

First, suppose that the deviating firm only targets stealing the incoming generation of new customers. When \( p_1 \in (0, U] \), clearly by undercutting \( p_1 \) by an infinitesimal amount, the deviating firm can attract all new customers. With our assumption that consumers are unsophisticated, when \( p_1 \in (u, (1 + \delta) U] \), it remains true that cutting \( p_1 \) by any infinitesimal amount allows the deviating firm to attract all new consumers.\(^{17}\) Besides cutting \( p_1 \), the deviating firm can also raise \( p_2 \), the price for its own repeat-purchase customers, to arbitrarily close to \( p_1 \) or \( U \) if \( p_1 > U \), knowing that they would have to pay \( p_1 \) if they buy from other firms. The profit from these customers is \( \min\{p_1, U\} \), leading to a deviation profit of \( p_1 + \frac{\min\{p_1, U\}}{n} \).

Alternatively, the deviating firm can undercut \( p_2 \) to attract all of the customers currently

\(^{17}\)If consumers correctly anticipated marginal cost pricing in the period following a deviation, no consumer would entertain a cut in \( p_1 \) unless it fell below \( U \) because consumers would be better off abstaining from consumption for one period.
in the market, resulting in a profit of 2\(p_2\). For collusion to be sustainable, we need
\[
\frac{p_1 + p_2}{n(1-\delta)} \geq \max\left\{ p_1 + \frac{\min\{p_1,U\}}{n}, 2p_2 \right\}. \tag{3}
\]

In the following proposition, we show that when \(p_1\) and \(p_2\) are properly selected, (3) is easier to satisfy than (1), its counterpart under uniform pricing.

**Proposition 1** Define \(\hat{\delta}^{NC} = -\frac{5n+1+\sqrt{16n^2-8n^4+17n^2-10n+1}}{4n^2}\). If \(\delta \in [\hat{\delta}^{NC},1)\), then any industry profit \(\Pi \in [0,1+\delta U]\) is sustainable as a SPNE in which firms reward loyalty without using commitment (NC). If \(\delta \in (0,\hat{\delta}^{NC})\), then there is no profitable SPNE in which firms reward loyalty without using commitment. Moreover, \(\hat{\delta}^{NC} < \hat{\delta}^U\).

**Proof.** If \(p_1 + \frac{\min\{p_1,U\}}{n} > 2p_2\), then the incentive constraint (3) becomes
\[
\frac{p_1 + p_2}{n(1-\delta)} \geq p_1 + \frac{\min\{p_1,U\}}{n}
\]
where the threshold \(\delta\) decreases in \(p_2\). So raising \(p_2\) to \(np_1 + \min\{p_1,U\}\) make collusion easiest.

Now, suppose \(p_1 + \frac{\min\{p_1,U\}}{n} < 2p_2\). The incentive constraint (3) becomes
\[
\frac{p_1 + p_2}{n(1-\delta)} \geq 2p_2
\]
where the threshold \(\delta\) increases in \(p_2\). In other words, collusion is easiest to sustain by lowering \(p_2\) to \(\frac{np_1 + \min\{p_1,U\}}{2n}\).

Summing up, no collusion at any price (or profit level) with \(p_1 \in (0,(1+\delta)U]\) is sustainable if (3) does not hold for \((p_1,p_2)\) satisfying \(p_1 + \frac{\min\{p_1,U\}}{n} = 2p_2\), i.e.,
\[
p_1 = \begin{cases} 
\frac{2n}{n+1}p_2 & \text{if } p_1 \leq U, \\
2p_2 - \frac{U}{n} & \text{if } p_1 > U.
\end{cases}
\]

If \(p_1 \leq U\), plugging \(p_1 = \frac{2n}{n+1}p_2\) into (3), we obtain the necessary and sufficient condition for tacit collusion to be sustainable:
\[
\frac{\frac{2n}{n+1}p_2 + p_2}{n(1-\delta)} \geq 2p_2
\]
\[
\delta \geq \delta_1 = 1 - \frac{3n + 1}{2n(n+1)}. \tag{4}
\]
When \( p_1 > U \), \( p_1 = 2p_2 - \frac{U}{n} > 2p_2 - \frac{p_1}{n} \), which implies \( p_1 > \frac{2n+1}{n+1}p_2 \). So \( \frac{p_1+p_2}{n(1-\delta)} \geq 2p_2 \) is sustainable for a larger range of \( \delta \) than in (4). The incentive constraint can be rewritten as

\[
\frac{2p_2 - \frac{U}{n} + p_2}{n(1-\delta)} \geq 2p_2
\]

\[
\delta \geq 1 - \frac{3p_2 - \frac{U}{n}}{2np_2}.
\] (5)

Note that \( 1 - \frac{3p_2 - \frac{U}{n}}{2np_2} \) decreases in \( p_2 \). With \( p_1 = 2p_2 - \frac{U}{n} \), the maximum \( p_2 \) achievable is obtained when the industry earns the monopoly profit, i.e., when

\[
2p_2 - \frac{U}{n} + \delta p_2 = (1 + \delta)U \Rightarrow p_2 = \frac{(n + n\delta + 1)U}{n(2 + \delta)}.
\]

If (5) is not satisfied with \( p_2 = \frac{(n + n\delta + 1)U}{n(2 + \delta)} \), in that case (4) also fails, then no positive profit is sustainable. However, if (5) is satisfied with \( p_2 = \frac{(n + n\delta + 1)U}{n(2 + \delta)} \), i.e.,

\[
\delta \geq 1 - \frac{3\frac{(n + n\delta + 1)U}{n(2 + \delta)} - \frac{U}{n}}{2n\frac{(n + n\delta + 1)U}{n(2 + \delta)}} = 1 - \frac{(3n + 3n\delta + 1 - \delta)}{2(n + n\delta + 1)n},
\] (6)

then the monopoly profit is sustainable and so is any lower profit by public randomization.

Since \( 1 - \frac{(3n + 3n\delta + 1 - \delta)}{2(n + n\delta + 1)n} \) decreases with \( \delta \), the above expression can be rewritten as

\[
\delta \geq \hat{\delta}_{NC} \equiv \frac{-5n + 1 + \sqrt{16n^4 - 8n^3 + 17n^2 - 10n + 1}}{4n^2}
\] (7)

and the preceding logic implies \( \hat{\delta}_{NC} < \delta_1 \). Summing up, if \( \delta \geq \hat{\delta}_{NC} \), any industry profit \( \Pi \in [0, \frac{1+\delta}{1-\delta}U] \) is sustainable; if \( \delta < \hat{\delta}_{NC} \), then no positive profit can be sustained.

Finally, compare \( \hat{\delta}_{NC} \) with \( \hat{\delta}U \) in Lemma 1:

\[
\hat{\delta}_{NC} - \hat{\delta}U < \delta_1 - \hat{\delta}U = \frac{1}{n} - \frac{3n + 1}{n - 2n(n + 1)} = -\frac{n - 1}{2n(n + 1)} < 0.
\]

According to Proposition 1, when compared with uniform pricing, rewarding loyalty without using commitment on the equilibrium path allows firms to sustain supranormal profits for the range of discount factors \([\hat{\delta}_{NC}, \hat{\delta}U]\), which is not possible when firms use uniform pricing on the equilibrium path. Although a firm has two options of deviation
(instead of just one as in the case of uniform pricing), note that the deviation profit \(\max\{p_1 + \frac{\min\{p_2, U\}}{n}, 2p_2\}\) is no more than \(\max\{p_1 + \frac{p_2}{n}, 2p_2\}\). So as long as \(p_2\) is not significantly lower than \(p_1\) to the extent that a deviating firm can benefit substantially from reneging on the discounted price for its repeat-purchase customers, the deviation profit is less than one period of industry profit \((p_1 + p_2)\). More precisely, if \(p_2 \in (\frac{p_2}{n}, p_1)\), then \(2p_2 < p_1 + p_2\) and \(p_1 + \frac{\min\{p_2, U\}}{n} \leq p_1 + \frac{p_2}{n} < p_1 + p_2\). Therefore, tacit collusion is easier to sustain in this case than in the case of uniform pricing.

4.3 Tacit Collusion Supported by Commitment to Repeat-Purchase Price

Suppose that on the equilibrium path firms charge \(p_1\) in the first period of a consumer’s market life and commit to charging the customer a fixed price of \(p_2 \in [0, p_1)\) for her repeat purchase in the next period. In any period \(t \geq 2\), just as in the previous subsection, each firm can earn a collusive profit of

\[
\frac{p_1 + p_2}{n(1 - \delta)} = \frac{p_2}{n} + \frac{p_1 + \delta p_2}{n(1 - \delta)}.
\]

The deviating firm can only attract new consumers by undercutting \(p_1\). Once again, this is true even when \(p_1 > U\) because of our assumption of unsophisticated consumers. Alternatively, the deviating firm can attract both new and old consumers by undercutting \(p_2\). Therefore, no firm has an incentive to deviate from the equilibrium prices if and only if

\[
\frac{p_1 + p_2}{n(1 - \delta)} = \frac{p_2}{n} + \frac{p_1 + \delta p_2}{n(1 - \delta)} \geq \max\left\{p_1 + \frac{p_2}{n}, 2p_2\right\}. \tag{8}
\]

We can see that (8) is very similar to (3), except that when firms commit to offering a future price to their returning customers, a firm deviating by just undercutting \(p_1\) is unable to raise its price to its measure \(1/n\) of old customers from \(p_2\) to \(p_1\) or \(U\) if \(p_1 > U\). As a result, the profit from such a deviation is lowered to \(p_1 + \frac{p_2}{n}\) from \(p_1 + \frac{\min\{p_2, U\}}{n}\). Therefore, the overall deviation profit is lower than that when firms reward loyalty without using commitment. This implies that tacit collusion is sustainable for a wider range of discount factors when firms offer loyalty rewards by committing to a discounted repeat-purchase price than when they make no such commitment. The following proposition formally proves this point.

**Proposition 2** Define \(\hat{\delta}^{RPP} = 1 - \frac{3n-1}{2n^2}\). If \(\delta \in [\hat{\delta}^{RPP}, 1)\), then any industry profit \(\Pi \in [0, 1 + \frac{\delta}{1 - \delta}U]\) is sustainable as a SPNE in which firms offer loyalty rewards by committing to a repeat-purchase price (RPP). If \(\delta \in (0, \hat{\delta}^{RPP})\), then there is no profitable SPNE in which firms offer loyalty rewards by committing to a repeat-purchase price. Moreover, \(\hat{\delta}^{RPP} < \hat{\delta}^{NC}\).
Proof. Subtracting $p_2/n$ from both sides of (8), the incentive constraint becomes

$$\frac{p_1 + \delta p_2}{n (1 - \delta)} \geq \max \left\{ p_1, \frac{2n - 1}{n} p_2 \right\}. \quad (9)$$

If $p_1 > \frac{2n - 1}{n} p_2$, then the incentive constraint can be rewritten as

$$\frac{p_1 + \delta p_2}{n (1 - \delta)} \geq p_1,$$

which is easier to satisfy for a higher $p_2$.

If $p_1 < \frac{2n - 1}{n} p_2$, then the incentive constraint becomes

$$\frac{p_1 + \delta p_2}{n (1 - \delta)} \geq \frac{2n - 1}{n} p_2,$$

which is easier to satisfy for a higher $p_1$. Therefore, the incentive constraint can be satisfied for some prices if and only if there exists $(p_1, p_2)$ satisfying $p_1 = \frac{2n - 1}{n} p_2$ and equation (9). That is,

$$\frac{2n - 1}{n} p_2 + \delta p_2 \geq \frac{2n - 1}{n} p_2 \quad \delta \leq \hat{\delta}_{RPP} \equiv 1 - \frac{3n - 1}{2n^2}.$$

To achieve the highest feasible profit, set

$$p_1 + \delta p_2 = \frac{2n - 1}{n} p_2 + \delta p_2 = (1 + \delta) U.$$

We can then obtain

$$p_2 = \frac{(1 + \delta) n}{(2 + \delta) n - 1} U, \quad p_1 = \frac{(1 + \delta) (2n - 1)}{(2 + \delta) n - 1} U.$$

Finally, $\hat{\delta}_{RPP} < \hat{\delta}_{NC}$ because at $\delta = \hat{\delta}_{RPP}$, (6) fails:

$$\hat{\delta}_{RPP} = 1 - \left( 1 - \frac{3n + 3n \hat{\delta}_{RPP} + 1 - \hat{\delta}_{RPP}}{2(n + n \hat{\delta}_{RPP})} \right) \frac{2(n + n \hat{\delta}_{RPP} + 1)n}{2(n + n \hat{\delta}_{RPP} + 1)n}$$

$$= 1 - \frac{3n - 1}{2n^2} - \left( 1 - \frac{3n + 3n \left( 1 - \frac{3n - 1}{2n^2} \right) + 1 - \left( 1 - \frac{3n - 1}{2n^2} \right)}{2 \left( n + n \left( 1 - \frac{3n - 1}{2n^2} \right) + 1 \right) n} \right)$$

$$= -\frac{n - 1}{n (4n^2 - n + 1)} < 0.$$
According to Proposition 2, committing to a discounted repeat-purchase price enables firms to sustain supranormal profits for the range of discount factors $[\hat{\delta}_{\text{RPP}}, 1 - 1/n)$, which is not possible when firms use uniform pricing. Unlike the case of price discrimination without using commitment, committing to a repeat-purchase price allows firms to earn supranormal profits for $\delta \in [\hat{\delta}_{\text{RPP}}, \hat{\delta}_{\text{NC}})$.

It is clear that offering loyalty rewards in the form of a discounted repeat-purchase price allows firms to charge new customers and repeat-purchase customers different prices just as in the case when loyalty rewards are offered without the use of a commitment. We also explained that the reason tacit collusion is easier when firms commit to a repeat-purchase price is that a deviating firm can no longer renege on the discounted price for its loyal customers in the period in which it undercuts its competitors. Next, we show that if the commitment is in the form of a repeat-purchase discount, tacit collusion is made even easier.

### 4.4 Tacit Collusion Supported by Commitment to Repeat-Purchase Discount

Now suppose that firms commit to offering their new customers a fixed repeat-purchase discount of $D$ dollars off the regular price so that in equilibrium $p_2 = p_1 - D$. In practice, this can be accomplished by firms issuing $D$-dollars-off coupons to their new customers for future use.$^{18}$ Recall that we impose the restriction that $p_2 \geq 0$ both on and off the equilibrium path. Without this constraint, collusion is easier to sustain. We will elaborate on this point at the end of this subsection.

In any period $t \geq 2$, the equilibrium profit for each firm, including the profit from current old customers, $p_2/n$, is

$$\pi_{\text{equ}} = \frac{p_1 + p_2}{n(1 - \delta)} = \frac{2p_2 + D}{n(1 - \delta)}.$$

The deviating firm can capture the entire measure one of new customers by undercutting $p_1 (= p_2 + D)$ by an infinitesimal amount. Due to the commitment to its loyal customers, the deviating firm also has to cut the price to them $p_2$, although only by the same infinitesimal

$^{18}$An alternative way to commit to a repeat-purchase discount is to use percentage-off coupons. We conjecture that the sustainability of tacit collusion using percentage-off coupons lies somewhere between the sustainability of tacit collusion using dollars-off coupons and that using only commitment to repeat-purchase prices. The reason that commitment to repeat-purchase discounts makes tacit collusion easier than commitment to repeat-purchase prices does is that the former forces a firm which undercuts the regular price to also lower its price to its repeat-purchase customers. Such effect is less pronounced in the case of percentage-off coupons than in the case of dollars-off coupons because in the former case the actual amount of discount (certain percentage of the regular price) decreases when the regular price decreases.
amount in this case. This will lead to a deviation profit of (or more precisely, arbitrarily close to)
\[ \pi_{\text{dev},1} = \frac{p_2}{n} + p_2 + D. \]

Alternatively, by undercutting \( p_2 \), the deviating firm can attract all consumers. In this case, \( (2 - 1/n) \) consumers pay \( p_2 \) but the firm has to lower the price to its old customers, who have coupons, by a discrete amount from \( p_2 \) to \( p_2 - D \) or to zero if \( p_2 - D < 0 \). This will lead to a deviation profit of
\[
\max\left\{ \left( 2 - \frac{1}{n} \right) p_2 + \frac{p_2 - D}{n}, \left( 2 - \frac{1}{n} \right) p_2 \right\}.
\]

Summing up, no firm has an incentive to deviate if and only if
\[
\frac{2p_2 + D}{n (1 - \delta)} \geq \max\left\{ \frac{p_2}{n} + p_2 + D, \frac{p_2 - D}{n} + \left( 2 - \frac{1}{n} \right) p_2, \left( 2 - \frac{1}{n} \right) p_2 \right\} \quad (10)
\]
or equivalently
\[
\pi_{\text{equ}} \geq \max\{\pi_{\text{dev},1}, \pi_{\text{dev},2}, \pi_{\text{dev},3}\},
\]
where \( \pi_{\text{dev},2} \equiv \frac{p_2 - D}{n} + \left( 2 - \frac{1}{n} \right) p_2 \), \( \pi_{\text{dev},3} \equiv \left( 2 - \frac{1}{n} \right) p_2 \).

If we compare these deviation profits to those when firms commit to repeat-purchase prices in equation (8), we can see that the deviation profit from undercutting \( p_1 \), namely \( \pi_{\text{dev},1} \), is the same for both types of commitments, but the deviation profit from undercutting \( p_2 \), namely \( \max\{\pi_{\text{dev},2}, \pi_{\text{dev},3}\} \), is smaller than the corresponding deviation profit in equation (8), due to the fact that when firms commit to repeat-purchase discounts, as the deviating firm lowers its regular price from \( p_1 \) to \( p_2 \), it also has to lower its price to its loyal customers. A firm’s deviation profit, the maximum of \( \pi_{\text{dev},1} \) and \( \pi_{\text{dev},2} \), is therefore (weakly) lower here, rendering tacit collusion easier to sustain when firms commit to repeat-purchase discounts than when they commit to repeat-purchase prices. The following proposition formalizes this point.

**Proposition 3** Define \( \hat{\delta}^{\text{RPD}} = 1 - \frac{3n+1}{2n^2+n+1} \). If \( \delta \in [\hat{\delta}^{\text{RDP}}, 1) \), then any industry profit \( \Pi \in [0, \frac{1}{1-\delta}U] \) is sustainable as a SPNE in which firms reward loyalty by committing to a repeat-purchase discount (RPD). If \( \delta \in (0, \hat{\delta}^{\text{RDP}}) \), then there is no profitable SPNE in which firms reward loyalty by committing to a repeat-purchase discount. Moreover, \( \hat{\delta}^{\text{RDP}} < \hat{\delta}^{\text{RPP}} \).
Proof. First, establish that

\[ \max \{\pi_{\text{dev},1}, \pi_{\text{dev},2}, \pi_{\text{dev},3}\} = \max \{\pi_{\text{dev},1}, \pi_{\text{dev},2}\}. \]

This is because whenever \( \pi_{\text{dev},3} > \pi_{\text{dev},2} \), which happens if and only if \( p_2 < D \),

\[ \pi_{\text{dev},1} - \pi_{\text{dev},3} = \frac{p_2}{n} + p_2 + D - \left(2 - \frac{1}{n}\right)p_2 = D - p_2 + \frac{2p_2}{n} > 0. \]

Next, it can be verified through direct calculation that

\[ \frac{\partial \pi_{\text{equ}}}{\partial p_2} = \frac{2}{n(1 - \delta)}, \quad \frac{\partial \pi_{\text{dev},1}}{\partial p_2} = 1 + \frac{1}{n}, \quad \frac{\partial \pi_{\text{dev},2}}{\partial p_2} = 2. \]

When \( \delta \in \left(\frac{n-1}{n+1}, 1 - \frac{1}{n}\right) \), we have \( \frac{\partial \pi_{\text{dev},1}}{\partial p_2} < \frac{\partial \pi_{\text{equ}}}{\partial p_2} < \frac{\partial \pi_{\text{dev},2}}{\partial p_2} \). In this case, \( \pi_{\text{equ}} \geq \max \{\pi_{\text{dev},1}, \pi_{\text{dev},2}\} \) is the easiest to satisfy when \( \pi_{\text{dev},1} = \pi_{\text{dev},2} \). To see this, suppose this is not true and \( \pi_{\text{dev},1} > \pi_{\text{dev},2} \). Then \( \pi_{\text{dev},2} \) is irrelevant, and a higher \( p_2 \) would increase \( \pi_{\text{equ}} \) faster than \( \pi_{\text{dev},1} \), thus making the collusion easier to sustain. So it is optimal to raise \( p_2 \) until \( \pi_{\text{dev},1} = \pi_{\text{dev},2} \). A similar argument implies that \( \pi_{\text{dev},1} < \pi_{\text{dev},2} \) is also suboptimal.

Next, \( \pi_{\text{dev},1} = \pi_{\text{dev},2} \) is equivalent to

\[ p_2 = \frac{n+1}{n-1}D. \]

Plugging \( p_2 = \frac{n+1}{n-1}D \) into \( \pi_{\text{equ}} \geq \max \{\pi_{\text{dev},1}, \pi_{\text{dev},2}\} \), we can obtain

\[ \frac{2 \frac{n+1}{n-1}D + D}{n(1 - \delta)} \geq \frac{n+1}{n} \frac{n+1}{n-1}D + D \]

\[ \delta \geq \hat{\delta}_{RDP} = 1 - \frac{3n+1}{2n^2 + n + 1}. \]

It can be easily verified that \( \hat{\delta}_{RDP} \in \left(\frac{n-1}{n+1}, 1 - \frac{1}{n}\right) \). Since the equilibrium profit in (10) increases in \( \delta \) but the deviation profit is independent of \( \delta \), it is clearly the case that (10) is more readily satisfied for higher \( \delta \). Therefore, the fact that tacit collusion is sustainable at \( \delta = \hat{\delta}_{RDP} \) implies that it is sustainable for all \( \delta \geq \hat{\delta}_{RDP} \). Also, the fact that tacit collusion is impossible to sustain for all \( \delta \in \left(\frac{n-1}{n+1}, \hat{\delta}_{RDP}\right) \) implies that tacit collusion is impossible for all \( \delta < \hat{\delta}_{RDP} \).

The highest feasible industry profit is achieved by setting

\[ p_1 + \delta p_2 = (1 + \delta)U, \]

\[ \frac{n+1}{n-1}D + D + \delta \frac{n+1}{n-1}D = (1 + \delta)U. \]

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We can then obtain
\[ D = \frac{(1 + \delta)(n - 1)U}{2n + n\delta + \delta}, p_2 = \frac{(1 + \delta)(n + 1)U}{2n + n\delta + \delta}, p_1 = \frac{2n(1 + \delta)U}{2n + n\delta + \delta}. \]

Finally,
\[ \hat{\delta}^{RPD} - \hat{\delta}^{RPP} = \frac{3n - 1}{2n^2} - \frac{3n + 1}{2n^2 + n + 1} \]
\[ = -\frac{(n - 1)^2}{2n^2(2n^2 + n + 1)} < 0. \]

The intuition for why committing to a repeat-purchase discount is more effective than committing to a repeat-purchase price in facilitating tacit collusion is as follows. When firms commit to a repeat-purchase price, they only commit to not raising the price for repeat-purchase customers when they cut the price for new customers. In contrast, when they commit to a repeat-purchase discount, each firm commits to further lowering its (already discounted) price for its repeat-purchase customers by the same amount by which it lowers the regular price. If a deviating firm cuts its regular price from \( p_1 \) to just below \( p_2 \) to capture all consumer, the extra discount it has to offer to repeat-purchase customers will be \( D = p_1 - p_2 \). The need to provide extra discounts to repeat-purchase customers following some deviations lowers the overall deviation profit and makes tacit collusion easier to sustain.

It is useful to point out that loyalty programs facilitate collusion through two channels. First, as being illustrated in the case of loyalty rewards without commitment, a deviating firm must choose whether to slightly undercut the price for new customers without attracting old customers of other firms, or set a sufficiently lower price to attract all consumers. In either case the deviating firm is unable to capture the entire one-period industry profit before losing all future profits. Second, as being illustrated in the case of loyalty rewards with commitment to repeat purchase discount, when a deviating firm undercuts its rivals’ regular prices, it also reduces the profitability of its own customer base.\(^{19}\)

To conclude this section, we briefly explain why, absent the restriction of \( p_2 \geq 0 \), tacit collusion will be easier to sustain. Suppose firms continue to set prices such that \( p_2 \geq 0 \) on the equilibrium path, even though they are not restricted to do so. The only impact of removing the restriction is that sometimes, after lowering the regular price, the deviating

\(^{19}\)This second channel has been identified in the literature using two-period models. For example, Caminal and Matutes (1990) also found that the deviation profit is lower when firms commit to repeat-purchase discount as compared to the case when firms commit to repeat-purchase price. However, in their setting, offering loyalty reward by committing to repeat-purchase discount is profitable but offering loyalty reward by committing to repeat-purchase price is unprofitable.
firm ends up charging its old customers a negative price instead of a price of zero. In other words, lifting this restriction weakly lowers the deviation profit, and thus, makes collusion weakly easier to sustain.

4.5 Graphical Comparison of Threshold Discount Factors

In the previous sections we derived the various threshold discount factors as functions of the number of firms in the market. The expressions are

\[
\hat{\delta}^U = 1 - \frac{1}{n}, \quad \hat{\delta}^{NC} = \frac{-5n+1+\sqrt{16n^4-8n^3+17n^2-10n+1}}{4n^2}, \quad \hat{\delta}^{RPP} = 1 - \frac{3n-1}{2n^2}, \quad \hat{\delta}^{RPD} = 1 - \frac{3n+1}{2n^2+n+1}.
\]

To provide a more concrete comparison of these threshold discount factors, we plot them in Figure 2 as functions of the number of firms \(n\).

![Figure 2: Comparison of Threshold Discount Factors](image)

From the figure, we can see that the threshold discount factor for tacit collusion with uniform pricing (\(\hat{\delta}^U\)) is significantly higher than the other threshold discount factors (\(\hat{\delta}^{NC}\), \(\hat{\delta}^{RPP}\) and \(\hat{\delta}^{RPD}\)), and this difference remains significant when the number of firms increases. The other three threshold discount factors for collusion with loyalty rewards (\(\hat{\delta}^{NC}\), \(\hat{\delta}^{RPP}\) and \(\hat{\delta}^{RPD}\)) are closer to each other, and the differences among them shrink when \(n\) increases. This illustrates the point that while the commitment to loyalty discounts and the type of commitment used affect the extent to which repeat-purchase discounts facilitate collusion, they do not affect the results qualitatively.

\footnote{In Figure 2, we treat the number of firms as if it were continuous, even though it is a integer.}
For any number of firms in the market, we have characterized the threshold discount factors above which tacit collusion is sustainable under different pricing schemes. We could equivalently derive the threshold numbers of firms below which tacit collusion is sustainable for each discount factor as well. It can be shown that these threshold numbers of firms are

\[ \hat{n}_U = \frac{1}{1 - \delta}, \hat{n}_{NC} = \frac{1 + 5\delta + \sqrt{8\delta^3 + 17\delta^2 + 23 + 9}}{4(1 - \delta^2)}, \hat{n}_{RPP} = \frac{3 + \sqrt{1 + 8\delta}}{4(1 - \delta)}, \hat{n}_{RPD} = \frac{2 + \delta + \sqrt{(7\delta + 2)(2 - \delta)}}{4(1 - \delta)}. \]

It is quite evident from Figure 2 and can be formally verified that

\[ \hat{n}_U < \hat{n}_{NC} < \hat{n}_{RPP} < \hat{n}_{RPD}. \]

5 Discussions

From our formal analysis, we can see that loyalty rewards facilitate tacit collusion whether firms use no commitment, commit to repeat-purchase prices, or commit to repeat-purchase discounts. In this section, we argue that our main result is also robust to certain changes in assumptions by considering the following alternative assumptions: sophisticated consumers, uncertain demand over time and differentiated products/heterogeneous consumers. The case of pricing below marginal cost is discussed at the end of Conclusion.

Sophisticated consumers In the formal analysis, we assumed that consumers are not sophisticated enough to anticipate a price war upon observing a price cut. An alternative perspective would be that rational consumers are able to anticipate the ensuing price war upon the observation of a deviation. In an earlier version of this paper, we adopted this alternative assumption. There, we found the same results qualitatively, namely, \( \hat{\delta}_{RPD} < \hat{\delta}_{PFC} < \hat{\delta}_{NC} < \hat{\delta}_U \). The only difference is that when consumers are sophisticated, tacit collusion becomes even easier when firms offer loyalty rewards, i.e., \( \hat{\delta}_{RPD}, \hat{\delta}_{PFC}, \) and \( \hat{\delta}_{NC} \) all become smaller, while \( \hat{\delta}_U \) remains unchanged. Below we explain the intuition as to why assuming sophisticated consumers enables us to obtain stronger results.

When firms offer loyalty rewards, they can set \( p_1 > U > p_2 \) as long as \( p_1 + \delta p_2 \leq (1 + \delta) U \). One important implication of introducing sophisticated consumers is that a deviating firm cannot steal any customers if it only slightly undercuts \( p_1 \) when \( p_1 > U \). Instead, it has to lower the price to \( U \) to steal any customers. This is because consumers rationally anticipate a price war after a deviation and will not buy from the deviating firm if the price is above their instantaneous utility \( U \). This reduces the deviation profit even further and renders tacit collusion as being sustainable for a wider range of discount factors, compared to the case when consumers are unsophisticated. On the other hand, under uniform pricing, it is
impossible to set \( p_1 > U \) because of the restriction of \( p_1 = p_2 \). Therefore, \( \hat{U} \) remains at \( 1 - 1/n \). This explains why assuming sophisticated consumers strengthens the result that loyalty rewards facilitate tacit collusion. Also note that introducing sophisticated consumers does not change the fact that commitment to repeat-purchase price and commitment to repeat-purchase discount make loyalty rewards more effective (relative to loyalty rewards without commitment) in facilitating tacit collusion. Therefore, it does not affect the ordering: \( \hat{\delta}^{RPD} < \hat{\delta}^{PFC} < \hat{\delta}^{NC} \).

Our formal analysis is conducted based on a highly stylized model in which all consumers and firms are identical and consumers know their future demand perfectly. Therefore, it is important to check whether our findings are robust to introduction of demand uncertainty and consumer and firm heterogeneity. These are what we do next.

**Uncertain demand over time** One natural way to introduce demand uncertainty is to assume that the expected value in the second period remains at \( U \) but it may be higher or lower. More specifically, the utility of a old consumer is \( U_H > U \) with probability \( \theta \) and is \( U_L < U \) with probability \( (1 - \theta) \) and \( U = \theta U_H + (1 - \theta) U_L \). In such a setting, the *ex-ante* expected life time utility from consumption remains at \( (1 + \delta) U \). We focus our discussion on the case when \( U_L \) is close to \( U \).

With uniform pricing, it is impossible for firms to make consumers pay \( (1 + \delta) U \) over their life time because to guarantee that all of them will purchase regardless of the realization of second period demand, firms ought to charge \( p \leq U_L \), which is less than \( U \). With loyalty rewards in place, charging consumers \( (1 + \delta) U \) becomes possible, by setting \( p_1 > U, p_2 \leq U_L \) and \( p_1 + \delta p_2 = (1 + \delta) U \). Not only that loyalty rewards allow firms to earn a higher profit, we conjecture that they can also facilitate tacit collusion more effectively just like in the case without demand uncertainty. The same logic applies here that when \( p_1 > p_2 \), the deviating firm is forced to either steal only new consumers at the higher price \( p_1 \) or charge a lower price \( p_2 \), and either way the deviator fails to steal one period of equilibrium profit before losing all the future profits.

**Differentiated products and heterogeneous consumers** We have assumed that consumers and firms are homogeneous. However, it is natural to ask whether loyalty rewards continue to facilitate tacit collusion in a more realistic setting where firms sell differentiated products and face downward sloping demands from heterogeneous consumers. To answer this question, we formally extend our setup (in the case of \( n = 2 \)) in the Appendix to one in which consumers with heterogeneous firm preferences are uniformly distributed on the Hotelling line. We prove that our results that loyalty rewards facilitate tacit collusion and that the use of commitments makes tacit collusion even easier to sustain readily generalize in
this setting. This is true because introducing consumer heterogeneity does not alter the fact that when firms raise \( p_1 \) by a small amount above the uniform price and lower \( p_2 \) according to keep the total profit unchanged, the deviating firm is forced to either focus on stealing new consumers or lowering the price significantly to steal both new and old consumers. Neither way allows it to capture the deviation profit under uniform pricing.

6 Conclusion

In this paper, we have shown that offering loyalty rewards can soften competition by facilitating tacit collusion and that this anticompetitive effect is even more pronounced when the loyalty rewards are implemented by committing to repeat-purchase prices or discounts (e.g., via coupons). We first established our findings in a dynamic framework of homogenous products and showed that these findings are robust to the use of commitment, the type of commitment used, and the market structure. The robustness of the collusion-facilitating role of loyalty rewards in our dynamic framework stands in sharp contrast to the sensitivity of the existing findings established in two-period frameworks.

We have discussed several plausible extensions of the model in Section 5 to check whether our results stand up to some variations in the assumptions and formally analyzed the extension to heterogeneous consumers and firms in the Appendix. There are also other potential benefits of using loyalty rewards apart from making tacit collusion easier. In discussing demand uncertainty, we discovered that loyalty rewards may help firms expand the feasible set of equilibrium profits. Loyalty programs may also discipline firms’ incentives to hold up their existing customers when these customers have to incur an exogenous switching cost upon switching brands. In other words, commitment to rewarding loyal customers may be particularly useful in the presence of exogenous brand-switching costs.

Our analysis predicts that the optimal loyalty discount always involves \( p_2 \) above marginal cost. One may argue that this does not match up very well with loyalty programs observed in practice which may allow consumers to receive the product for free. For example, coffee shop loyalty programs offer a free coffee after 10 purchases and airline companies offer a free round-trip ticket after a frequent flier has accumulated certain number of frequent flier miles.\(^{21}\) We do not think that such observations contradict our prediction of a more moderate discount. Notice that in these examples consumers have to make multiple purchases before they qualify for a unit of free product, so the average reward per purchase is only a fraction of

\(^{21}\)Note that when frequent fliers redeem their award tickets using miles, they still have to pay relevant fees and taxes.
the value of the purchase. Viewed in this light, this is in line with what our analysis predicts. It is also quite common for frequent fliers who do not have enough frequent flier miles for an award ticket to use the “miles plus cash” option to redeem the ticket or they may simply use cash to purchase the miles that they fall short of. In those cases, passengers actually have to pay a significant portion of the original price of a ticket. In practice, firms may favor giving a significant reward to a customer after several purchases over giving a moderate reward after each purchase to save on hassle cost and/or to further solidify customer loyalty. We do recognize that to properly evaluate the performance of reward programs which let consumers receive a free unit after multiple purchases requires us to extend our model to one in which consumers have more than two periods of market lives. We leave this to future research.

Appendix: Differentiated Products and Heterogeneous Consumers

Model There are two infinitely-lived firms, A and B, producing differentiated products. Firms A and B are located at point 0 and point 1 on the [0, 1] Hotelling line, respectively. Each firm has a constant marginal cost, which we normalize to 0. Consumers arrive in overlapping generations. In each period, there is a continuum of consumers of measure one entering the market and each consumer lives for two periods. They are uniformly distributed on the interval [0, 1] and each consumer’s location remains unchanged across periods. In other words, her firm preference does not vary over time. Consumers are semianonymous.

Let $U$ be the consumers’ use value of either of the firm’s products and $t$ be the unit transportation cost. A consumer located at $x$ enjoys a utility $u(A) = U - p_A - tx$ if she buys from firm A at price $p_A$. If she buys from firm B, her utility is $u(B) = U - p_B - t(1 - x)$. We assume that $U$ is sufficiently high, namely $U > 3t/2$, so that the market is covered in the static Nash equilibrium. Firms and consumers have a common discount factor, $\delta \in (0, 1)$. It is well known that the static Nash equilibrium price is $p = t$.

Analysis In this section, we formally compare firms’ abilities to tacitly collude in the cases when firms use the following pricing schemes: (1) uniform pricing and (2) rewarding loyalty without using commitment. The next proposition shows that the latter pricing scheme makes collusion easier to sustain.

**Proposition A1** Consider any collusive uniform price $p \in (t, U - t/2]$. There exists $\epsilon > 0$ such that by charging repeat-purchase customers $p_2 = p - \epsilon$ and all other customers $p_1 = p + \delta \epsilon$, firms A and B achieve the same industry profit for a wider range of discount factors.

An immediate corollary of Proposition A1 is that offering loyalty rewards by committing to offering either a repeat-purchase price or a repeat-purchase discount also facilitates tacit
collusion when firms produce differentiated products and consumers have heterogeneous firm preferences. This is because, compared to loyalty rewards without commitment, commitment to a fixed repeat-purchase price or repeat-purchase discount prevents a deviating firm from raising the price for its own old customers, and, thus, lowers deviation profits, making tacit collusion even easier to sustain.

The basic idea behind the proof of Proposition A1 is as follows. When firms tacitly collude using uniform pricing to raise the equilibrium price, firms may be tempted to undercut the inflated price. An alternative pricing strategy facilitates tacit collusion if it gives rise to the same equilibrium profit but lowers the deviation profit. Since a deviating firm lowers its price when it deviates, in the period it deviates, it will retain the entire measure one of consumers who would purchase from it in equilibrium. When arguing that loyalty rewards without commitment facilitate tacit collusion, the key is to show that when firms reward loyalty in equilibrium, the total deviation demand coming from the measure one of customers closer to the competitor becomes lower. Note that half of these consumers are new and the other half previously purchased from the competitor. Suppose the initial uniform price is $p_1 = p_2 = p$ and the firms then raise $p_1$ to $p + \delta \varepsilon$ and lower $p_2$ to $p - \varepsilon$, so that the total price each consumer pays over her lifetime stays unchanged at $p_1 + \delta p_2 = (1 + \delta) p$. Since firm $B$ charges its loyal customers $p - \varepsilon$, those customers’ willingness to pay for firm $A$’s product is reduced by $\varepsilon$, and similarly since firm $B$ charges new consumers $p + \delta \varepsilon$, the new customers’ willingness to pay for firm $A$’s product is increased by $\delta \varepsilon$. Given any deviation in price, the total demand coming from these two groups of consumers changes by $\frac{\delta \varepsilon}{2t} - \frac{\varepsilon}{2t} = -\frac{(1-\delta)\varepsilon}{2t}$. Due to the drop in the overall demand from these consumers, loyalty rewards lower the deviation profit, making tacit collusion easier to sustain, as in the case of homogeneous consumers.

**Proof of Proposition A1.** Let $p \in (t, U - t/2]$ denote the collusive price under uniform pricing. Without loss of generality, assume that firm $A$ is the deviating firm with $p_{dev}$ being its deviation price and $q_{dev}(p_{dev})$ being its deviating demand.$^{22}$ Consider any period $t \geq 2$. There is a measure one of new customers and a measure one of old customers. For ease of exposition, we ignore the consumers’ outside option of not purchasing from either firm. This outside option may be relevant if consumers prefer it to buying from firm $B$. The omission of this outside option will overestimate firm $A$’s deviation demand when $p_{dev}$ is sufficiently higher than $p$ and sufficiently close to consumers’ use value $U$. We would, however, show that such omission does not affect the validity of our proof.

$^{22}$The firm will not charge its old customers a higher price because then they will not reveal their identity but pretend to be new customers instead. The firm has no incentive to charge its own old customers a lower price either because they have stronger preference for its product. For this reason, there is no loss of generality in assuming that a deviating firm chooses only one price.
When $p_{\text{dev}} > p + t$, 
\[ p_{\text{dev}} + tx > p + t(1 - x) \]
for all $x \geq 0$; so the deviator’s demand is zero. When $p_{\text{dev}} < p - t$, 
\[ p_{\text{dev}} + tx < p + t(1 - x) \]
for all $x \leq 1$; so the deviator captures the entire measure two of customers. When $p_{\text{dev}} \in [p - t, p + t]$, let $\bar{x}$ denote the marginal consumer who is indifferent between buying from firm A at $p_{\text{dev}}$ and buying from firm B at $p$. Then, 
\[ p_{\text{dev}} + t\bar{x} = p + t(1 - \bar{x}) \]
and the deviator’s demand is 
\[ q_{\text{dev}} = 2\bar{x} = \frac{p + t - p_{\text{dev}}}{t}. \] (11)
Figure A1 illustrates firm A’s deviation demand under uniform pricing.

The corresponding deviation profit, denoted by $\pi_{\text{dev}}$, is 
\[ \pi_{\text{dev}} = \begin{cases} 
0 & \text{if } p_{\text{dev}} > p + t, \\
p_{\text{dev}} \frac{p + t - p_{\text{dev}}}{t} & \text{if } p_{\text{dev}} \in [p - t, p + t], \\
2p_{\text{dev}} & \text{if } p_{\text{dev}} < p - t.
\end{cases} \]
It can be readily checked that the optimal deviation price, denoted by $p_{\text{dev}}^*$, is strictly less than $p$: 
\[ p_{\text{dev}}^* = \max \left\{ \frac{p + t}{2}, p - t \right\} < p. \]
Since the deviator chooses $p^*_\text{dev} < p$ even when the deviation profit may be overestimated for $p_{\text{dev}} > p$, when the overestimation is corrected the optimal deviation price should remain at $p^*_\text{dev}$ and the optimal deviation profit is unaffected.

Next we consider the case of using loyalty rewards without commitment. Let $p_1 = p + \delta \epsilon, \ p_2 = p - \epsilon$. Firms charge $p_2$ to their own old customers, and $p_1$ to the rest of the customers. Then offering loyalty rewards with $(p_1, p_2)$ leads to the same collusive industry profit, $p (1 + \delta)$ from each generation of customers, as that under uniform pricing $(p, p)$. Next we want to show that as $\epsilon \to 0^+$, offering loyalty rewards necessarily makes collusion easier to sustain than uniform pricing, i.e., $\epsilon = 0$. Note that, since firms reward loyalty without using commitment, a deviating firm can choose a deviation price above $p_2$ and charge this price to its own old customers.

Under uniform pricing, the deviation demand is well-behaved and has no kinks. However, when firms reward loyal customers, the demand curve has several kinks. This is because the deviation demand for different groups of customers takes different shapes depending on whether they are new or old customers, and if they are old, which firm they bought from. In particular, the customers can be divided into two groups: (1) firm $B$’s old customers (facing $p_2$ from firm $B$) and (2) firm $A$’s old customers and new customers (facing $p_1$ from firm $B$). There is a measure one of new customers distributed on the interval $[0, 1]$. There is a measure one of old customers also distributed on the interval $[0, 1]$. Of the old customers, those on $[0, 1/2]$ bought from firm $A$ before, and those on $[1/2, 1]$ bought from firm $B$. Figure A2 illustrates firm $A$’s deviation demand by customer groups, and the overall deviation demand which is the kinked demand curve $ABCDE$.

Again for ease of exposition, in both Figure A2 and the following calculations, we ignore the consumers’ outside option of not purchasing at all when some consumers prefer this option to purchasing from the non-deviating firm. We also ignore the consideration of new consumers closer to firm $B$, i.e., those in $(1/2, 1]$, that by buying from the deviating firm $A$ instead of $B$, they will enjoy less utility in the follow period. Both omissions lead to overestimation of the deviating firm’s demand. We will show that such omissions do not affect the validity of the proof.

**Group 1: Firm $B$’s old customers**

A measure $1/2$ of these customers is uniformly distributed on the interval $[1/2, 1]$. They

\footnote{Recall the assumption unsophisticated consumers expect prices to return to the equilibrium level after deviation. Therefore, regardless of which firm they buy from in the deviation period, they will face the same $p_2$ from that firm in the following period. But they are further away from firm $A$ than firm $B$.}
will pay \( p_2 = p - \epsilon \) if they buy from firm \( B \). Let \( x_1 \) denote the marginal consumer. Then

\[
p_{\text{dev}} + tx_1 = p_2 + t(1 - x_1) \Rightarrow x_1 = \frac{p + t - \epsilon - p_{\text{dev}}}{2t}.
\]

Firm \( A \)'s deviation demand from these customers is \( x_1 - 1/2 \). When \( p_{\text{dev}} \geq p - \epsilon, x_1 \leq 1/2 \), firm \( A \)'s deviation demand is 0. When \( p_{\text{dev}} \leq p - \epsilon - t, x_1 \geq 1 \), Firm \( A \) sells to all firm \( B \)'s old customers and its deviation demand is 1/2.

**Group 2: New customers and firm \( A \)'s old customers**

All of these customers face \( p_1 = p + \delta \epsilon \) if they buy from firm \( B \). A measure 1 of new customers is uniformly distributed on the interval \([0,1]\), and a measure 1/2 of firm \( A \)'s old customers on the interval \([0,1/2]\).
Let \( x_2 \) denote the marginal customer. Then

\[
p_{\text{dev}} + tx_2 = p_1 + t(1 - x_2) \Rightarrow x_2 = \frac{p + t + \delta \epsilon - p_{\text{dev}}}{2t}.
\]

When \( p_{\text{dev}} \geq p + t + \delta \epsilon, x_2 \leq 0 \) and firm A’s demand is zero. When \( p_{\text{dev}} \in [p + \delta \epsilon, p + t + \delta \epsilon) \), \( x_2 \in (0, 1/2) \), and firm A’s deviation demand is \( 2x_2 \). When \( p_{\text{dev}} \in (p - t + \delta \epsilon, p + \delta \epsilon] \), \( x_2 \in (1/2, 1) \). In this case, firm A sells to all its own old customers (measure 1/2) and part of the new customers. Its deviation demand is \( \frac{1}{2} + x_2 \).

When \( p_{\text{dev}} \leq p - t + \delta \epsilon, x_2 > 1 \). Firm A sells to all of these customers and its demand is \( 3/2 \) (measure 1/2 for old and measure 1 for new).

We can then aggregate firm A’s deviation demands from customers in Group 1 and Group 2 to obtain firm A’s total deviation demand.

By continuity, as \( \epsilon \to 0 \), the optimal deviation price converges to the uniform pricing level \( p^*_{\text{dev}} \), which is strictly less than \( p \). This implies that there exists \( \bar{\epsilon} > 0 \) such that for all \( \epsilon < \bar{\epsilon} \), \( \hat{p}^*_{\text{dev}} < p - \epsilon \), where \( \hat{p}^*_{\text{dev}} \) denotes the optimal deviation price when firms offer loyalty rewards. Let \( \hat{q}_{\text{dev}}(p_{\text{dev}}) \) denote the deviation demand function when firms offer loyalty rewards. For \( p_{\text{dev}} \in [p - t + \delta \epsilon, p - \epsilon) \),

\[
\hat{q}_{\text{dev}} = (x_1 - 0.5) + 0.5 + x_2 = \frac{p + t - \epsilon - p_{\text{dev}}}{2t} + \frac{p + t + \delta \epsilon - p_{\text{dev}}}{2t} < \frac{p + t - p_{\text{dev}}}{t} = q_{\text{dev}}.
\]

For \( p_{\text{dev}} \in [p - t - \epsilon, p - t + \delta \epsilon) \),

\[
\hat{q}_{\text{dev}} = (x_1 - 0.5) + 3/2 = \frac{p + t - \epsilon - p_{\text{dev}}}{2t} + 1 < \frac{p + t - p_{\text{dev}}}{2t} + \frac{p + t + \delta \epsilon - p_{\text{dev}}}{2t} < q_{\text{dev}},
\]

where the first inequality follows immediately \( p_{\text{dev}} < p - t + \delta \epsilon \). Let \( \hat{\pi}^*_{\text{dev}} \) denote the maximum deviation profit when firms offer loyalty rewards. Together, \( \hat{p}^*_{\text{dev}} < p - \epsilon \) and \( \hat{q}_{\text{dev}} < q_{\text{dev}} \) imply that \( \hat{\pi}^*_{\text{dev}} < \pi^*_{\text{dev}} \). Since the above \( \hat{\pi}^*_{\text{dev}} \) is overestimated, the actual \( \hat{\pi}^*_{\text{dev}} \) must also be less than \( \pi^*_{\text{dev}} \). Since the loyalty rewards without commitment leads to lower deviation profit with the same collusive profit, it must be that collusion can be sustained under a wider range of discount factors, relative to the case of uniform prices. ■
References


