Web Appendix to “When Does Aftermarket Monopolization Soften Foremarket Competition?”

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Abstract

In this document, we (i) characterize the equilibrium payoff set assuming that consumers are unsophisticated in the sense that they are unable to anticipate a price war upon the observation of a firm’s deviation, (ii) extend the main model to allow consumers to stay in the market for more than two periods, and (iii) show that if commitment to future prices is inefficient because demand for cartridge is uncertain and cartridge quality is unverifiable, then even if firms can (inefficiently) commit to future prices, supranormal industry profit is still sustainable among arbitrarily many firms.

1 Full Characterization: The Case of Unsophisticated consumers

In this section of the web appendix, we characterize the number of firms among which tacit collusion is sustainable for any profit level, assuming that consumers are unsophisticated in the sense that they are unable to understand the collusive pricing strategy firms use to sustain the equilibrium prices. More specifically, we assume that because consumers are unaware of firms’ collusive pricing strategies, regardless of the equilibrium $P$ and $p$, when they observe any price deviation, they believe that next period, firms will continue to charge the equilibrium prices. Recall that by contrast, when consumers

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were sophisticated enough to anticipate a price war upon seeing a price cut, for \( P > U \), consumers would not entertain a price cut with \( P' \in (U, P) \). Under the alternative assumption in this section, any deviation with \( P' < P \) will attract customers and deviation profit will be (weakly) higher. We will show that despite this change, the analysis is qualitatively very similar to that in the paper.

Our main objective in this extension is to identify, for all \( \delta \in (0, 1) \) and for all \( n \geq 2 \), the range of steady state per-generation industry profits that can be supported by tacit collusion. Just like in the analysis in the paper, we assume that following any deviation from a collusion outcome, firms revert forever to a SPE play path in which they earn zero profits from each new generation of consumers, except here consumers fail to anticipate this when they see a deviation.

1.1 Punishment Path: Zero-Profit Equilibrium

The zero profit equilibrium is unaffected by the assumption of unsophisticated consumers. First, in any equilibrium in which firms earn zero profit from each consumer’s life-cycle demands, \( p = P \) must continue to hold for the following reasons. Suppose \( p > P \). Then no cartridges would be sold and the per-generation industry profit would be \( \pi = P - C \) and zero profit would imply \( P = C \). In this case, a firm could earn a positive profit by lowering its cartridge price to some \( p' \in (c, C) \) just like before. Next, if \( p < P \), then a firm could raise its profit above zero by charging its established customers a higher price \( p'' \in (p, P) \) for the cartridge, also just like before. The deviating firm’s established customers will continue to purchase its cartridge because a new printer costs more.

Furthermore, in any zero-profit equilibrium all established customers purchase a compatible cartridge. Since \( p = P \), if some established customers purchased new printers and firms broke even, then some firms could earn a positive profit by lowering the cartridge price by an infinitesimal amount to induce these established customers to purchase the cartridge instead of the printer. Let \( p^C \) denote the common price in a zero-profit equilibrium. Then \( (p^C - C) + \delta (p^C - c) = 0 \), or

\[
    c < p = P = p^C = C + \delta c 
\]

(1)

It is easy to verify that no firm has an incentive to deviate if they believe that following the deviation all other firms will continue to charge the equilibrium prices. Let the deviating firm be called Firm A. One period after deviation, firm A will be the only firm which has established customers. It is indeed an equilibrium that every firm continues to charge the zero-profit equilibrium prices. Firm A once again has no incentive to steal others’ business given its belief that in the following period, other firms’ prices
remain the same. If it did that, it would earn less from the established customers it stole and earn a negative from yet another generation of new consumers. Any other firm also will not deviate under the zero-profit prices. Doing so will cause it to capture A’s new customers, causing it to make a loss.

1.2 The Most Effective Collusive Prices

For most parts, we can directly borrow from the characterization in the case of sophisticated consumers. For instance, all the on-the-equilibrium path behaviors and the associated beliefs are exactly the same as before. The only places the assumption of unsophisticated consumers makes a difference is on the off-the-equilibrium paths. In fact, when \( P \leq U \), since whether consumers are sophisticated or unsophisticated, any deviation price \( P' \) can attract all new consumers. Therefore, we only need to derive the deviation payoff when \( P > U \) for the case of unsophisticated consumers. All other expressions will be borrowed from the case of sophisticated consumers. To make this section self-contained, however, we will repeat many of the steps in the paper.

To characterize the set of equilibrium profits sustainable by tacit collusion, we define the most effective collusive prices the same way we defined it in the paper. Similarly, WLOG, we assume that firms always adopt the most effective collusive prices.

Also similar to the main model, the following should hold for the same argument:

**Lemma AA1**  The most effective collusive prices must satisfy \( p \leq P \).

**Proof.** Suppose instead that \( p > P \) in equilibrium. Every established customer would strictly prefer purchasing a new printer to purchasing a replacement cartridge. So no cartridges would be sold and the per-generation industry profit would be \( \pi = (P - C) + \delta (P - C) \). By cutting the printer price below \( P \), a deviating firm could steal all the new and established customers.

Suppose firms instead coordinate on the common printer and cartridge price \( p' \), where

\[
p' = P - \frac{\delta (C - c)}{1 + \delta} < P < p,
\]

so that cartridges will be sold in equilibrium. One can verify that the equilibrium per-generation industry profit would remain at \( (P - C) + \delta (P - C) \). However, it would now require a deviating firm to cut the printer price below \( p' \) to steal both the new and established customers. This would lower the deviation profit and weaken the incentives to deviate.

\[\blacksquare\]
Just like before, it is necessary that \( p \leq U \) and \( P \leq U + \delta(U - p) \) for established customers to buy the cartridge and new consumers to buy the printer.\(^1\) Suppose firms collude on the price pair \((P,p)\) such that the industry is earning a profit of \( \pi \equiv (P - C) + \delta(p - c) > 0 \) per generation of customers. In the steady state, each firm will earn a discounted profit of

\[
\frac{\pi}{n(1 - \delta)} = \frac{(P - C) + \delta(p - c)}{n(1 - \delta)}
\]

from customers entering the market in the current and all the future periods. Note that a profit of \((p - c)/n\) which comes from the established customers who already purchased the printer in the previous period is excluded from this expression.

Now consider a firm’s deviation payoff. First, look at the case where \( p < P \). Since consumers are unsophisticated, they cannot anticipate that both the printer and cartridge prices will become \( \frac{C + c}{1 + \delta} \) in the period following a unilateral deviation. Instead, they believe that the cartridge price will stay at the equilibrium level. With such off-the-equilibrium path belief, the deviating firm can attracts an entire generation of new customers by setting a printer price \( P' \) arbitrarily close to but less than \( P \). Because consumers are unsophisticated, this remains true even if \( P > U \), unlike in the case when consumers were sophisticated.

Since \( p \) is less than both \( P \) and \( U \), the deviating firm can also simultaneously raise the cartridge price up to \( \min\{P',U\} \) without losing its measure \( 1/n \) of established customers. This leads to an instantaneous deviation profit arbitrarily close to

\[
(P - C) + \frac{\min\{P,U\} - c}{n}.
\]

The main difference from the case of sophisticated consumers is that there, when \( P > U \), the first term of the deviation profit would be \( \min\{P,U\} - C \). It is no longer true here.

Just like before, if the firm cuts the printer price further so that \( P' \) is arbitrarily close to but below \( p \), then it also attracts a measure \((n - 1)/n\) of established customers from its competitors. By doing so, it will earn an instantaneous profit arbitrarily close to \((2n - 1)(p - C)/n\). Note that the deviating firm has to lower its cartridge price (by an infinitesimal amount) to \( P' \) as well to prevent its existing customers from replacing its old printer with a new one. However, since \( P' \) is arbitrarily close to \( p \), the latter price cut does not affect the deviation profit.

\(^1\)\text{Note that while we assume here that consumers are unable to anticipate a price war upon observing a deviation, we still assume that consumers are forward looking and that they maximize life-time utility.}
Whether the deviating firm undercuts $P$ or $p$, the new consumers it attracts now will continue to purchase the cartridge from it at the price of $p^C = (C + \delta c) / (1 + \delta)$ when they are old in the following period, allowing it to earn an additional discounted profit of $\delta (C - c) / (1 + \delta)$. The deviating firm does not earn additional profits from the competitors’ existing customers it has attracted because they will leave the market in the following period.

Due to the ensuing price war, the deviating firm will not earn any more profit from future generations of customers. This gives rise to the following incentive constraint for firms to stay collusive:\(^1\)

$$\frac{p - c}{n} + \frac{(P - C) + \delta (p - c)}{n(1 - \delta)} \geq \max \left\{ (P - C) + \frac{\min \{P, U\} - c}{n} + \delta \frac{(C - c)}{1 + \delta}, \frac{p - c}{n} + \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta} \right\}, \text{ for } p < P$$

or

$$\frac{(P - C) + \delta (p - c)}{n(1 - \delta)} \geq \max \left\{ (P - C) + \frac{\min \{P, U\} - p}{n} + \delta \frac{(C - c)}{1 + \delta}, \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta} \right\}, \text{ for } p < P. \quad (2)$$

The only difference of this incentive constraint from the corresponding one in the case of sophisticated consumers lies in the first RHS term, replacing $\min \{P, U\} - C$ with simply $(P - C)$. This will somewhat simplifying the characterization of the most effective collusive prices and the equilibrium profit sets or the maximum number of firms among which each profit level is sustainable.

Next, look at the case where firms collude by setting $P = p \leq U$. In this case, the incentive constraint is exactly the same as its counter part in the case of sophisticated consumers, namely,

$$\frac{(P - C) + \delta (p - c)}{n(1 - \delta)} \geq \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta}, \text{ for } p = P. \quad (3)$$

To most effectively collude, for any given per-generation industry profit level $\pi$ that the firms target to achieve, firms choose a price pair $(P, p)$ satisfying $(P - C) + \delta (p - c) = \pi$ such that the deviation profit is minimized. This transforms the identification of the most effective collusive prices into the following problem of minimizing a firm’s deviation payoff:

$$\min_{(p, P)} D = \begin{cases} \max \left\{ (P - C) + \frac{\min \{P, U\} - p}{n} + \delta \frac{(C - c)}{1 + \delta}, \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta} \right\} & \text{if } p < P, \\ \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta} & \text{if } p = P, \end{cases}$$

subject to $(P - C) + \delta (p - c) = \pi$, $p \leq P$. \quad (4)

The following proposition characterizes the most effective collusive prices that solves Problem (4):

\(^2\)The profit from the established customers who already purchased the printer in the previous period is excluded from both sides of the inequality sign.
Proposition AA1 Suppose $U > C$ so that $\delta (C - c) < \pi^M$. The most effective collusive prices are

$$(P, p) = \begin{cases} 
(\pi + C - \delta (\hat{p}_{13} - c), \hat{p}_{13}) & \text{if } \pi \in (\frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c, \pi^M], \\
(\pi + C - \delta (\hat{p}_{23} - c), \hat{p}_{23}) & \text{if } \pi \in (\delta (C - c), \frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c], \\
(\hat{p}, \hat{p}) & \text{if } \pi \leq \delta (C - c). 
\end{cases}$$

where

$$\hat{p}_{13} = \frac{U - C + n \pi + n (2C + \delta c)}{(2 + \delta) n}$$

$$\hat{p}_{23} = \frac{(n + 1) \pi + 2nC + (n + 1) \delta c}{2n + n\delta + \delta}.$$

**Proof.** By substituting the rearranged constraint

$$P = \pi + C - \delta (p - c)$$

into the minimization problem (4), the latter can be rewritten as:

$$\min_{p \in [p^C, \bar{p}]} D(p, \pi) = \begin{cases} 
\max \{ \min \{ g_1(p, \pi), g_2(p, \pi) \}, g_3(p) \} & \text{if } p < \bar{p}, \\
g_3(p) & \text{if } p = \bar{p}, 
\end{cases}$$

where $\bar{p} \equiv \frac{\pi + C + \delta c}{1 + \delta}$ and

$$\min \{ g_1, g_2 \} = \begin{cases} 
g_1(p, \pi) & \text{if } p < \frac{\pi + U + \delta c}{\delta}, \\
g_2(p, \pi) & \text{if } p \in [\frac{\pi + U + \delta c}{\delta}, \bar{p}], 
\end{cases}$$

$$g_1 = \pi - \delta (p - c) + \frac{U - p}{n} + \delta \frac{(C - c)}{1 + \delta},$$

$$g_2 = \pi - \delta (p - c) + \frac{\pi + C - \delta (p - c) - p}{n} + \delta \frac{(C - c)}{1 + \delta},$$

$$g_3(p) = \frac{(2n - 1)(p - C)}{n} + \delta \frac{(C - c)}{1 + \delta}.$$

It can be directly verified that

$$\frac{\partial g_2}{\partial p} = -\frac{\delta + n\delta + 1}{n} < -\frac{n\delta + 1}{n} = \frac{\partial g_1}{\partial p} < 0 < \frac{2n - 1}{n} = \frac{\partial g_3}{\partial p}.$$  

(7)

Let $p = \hat{p}_{12}$ solve $g_1 = g_2$, $p = \hat{p}_{13}$ solve $g_1 = g_3$, and $p = \hat{p}_{23}$ solve $g_2 = g_3$. It can be verified that

$$\hat{p}_{12} = \frac{\pi - U + C + \delta c}{\delta},$$

$$\hat{p}_{13} = \frac{U - C + n \pi + n (2C + \delta c)}{(2 + \delta) n}$$

$$\hat{p}_{23} = \frac{(n + 1) \pi + 2nC + (n + 1) \delta c}{2n + n\delta + \delta}.$$
It can also be verified that $\hat{p}_{23} \leq \hat{p}$ if and only if
\[ \pi \geq \delta (C - c). \]

And $\hat{p}_{23} \leq \hat{p}_{12}$ if and only if
\[ \pi \geq \hat{\pi} = \frac{1}{2} \left( \frac{2n + \delta + n\delta}{n} U + \frac{1 - 2n - \delta + n\delta}{n} C - \delta c \right). \]

Summing up,
\[ \arg \min_{p \in [\hat{p}^c, \hat{p}]} D(p, \pi) = \begin{cases} \hat{p}_{13} & \text{if } \pi \in \left( \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{-2n - \delta + n\delta}{n} C - \delta c, \pi^M \right], \\ \hat{p}_{23} & \text{if } \pi \in \left( \delta (C - c), \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{-2n - \delta + n\delta}{n} C - \delta c \right], \\ \hat{p} & \text{if } \pi \leq \delta (C - c). \end{cases} \]

The most effective collusive $P$ is obtained by substituting these into (5).

1.3 Characterization of Profits Sustainable by Tacit Collusion

Building on Proposition AA1, we proceed to characterize the set of per-generation profits sustainable by tacit collusion.

Since $(U - C) + \delta (U - C) > \delta (C - c)$ if and only if $U > C$ and $(U - C) + \delta (U - C) > \frac{\delta (C - c)}{\delta}$ if and only if $U \geq \frac{(2\delta)(C + \delta c)}{2(1+\delta)}$, we have

\[ \pi^M > \delta (C - c) \quad \text{if } U > C, \]
\[ \pi^M \in \left( \frac{\delta (C - c)}{2}, \delta (C - c) \right] \quad \text{if } U \in \left( \frac{\delta (C - c)}{2(1+\delta)}, C \right], \]
\[ \pi^M \in \left( 0, \frac{\delta (C - c)}{2} \right] \quad \text{if } U \in \left( \frac{C + \delta c}{1+\delta}, \frac{(2\delta)(C + \delta c)}{2(1+\delta)} \right]. \]  

The following theorem characterizes collusive profits in the more complicated case when $U > C$.

**Theorem AA1** Suppose $C < U$. If

\[ \delta \geq \frac{3(C - c) + \sqrt{9(C - c)^2 + 16(U - C)[6(C - c) + 4(U - C)]}}{4[3(C - c) + 2(U - C)]}, \]  

then $\tilde{n}_3 < \tilde{n}_1 < \tilde{n}_2$. Any profit $\pi \in [0, \pi^M]$ can be sustained for all $n \leq \tilde{n}_3$.

For $n \in (\tilde{n}_3, \tilde{n}_1]$, the set of sustainable profit is

\[ \left[ 0, \frac{1 - \delta}{3 - 2n(1-\delta)} (2n-1)(1+\delta)(U-C)-\delta n((n-1)\delta-1)(C-c) \right] \]
If \( n \in (\tilde{n}_1, \tilde{n}_2] \), the set of sustainable profit is
\[
\left[ 0, \frac{(n-1)(1-\delta)((n+1)\delta+1)\delta(C-c)}{(1+\delta)(2(1-\delta)\gamma-1)(1+2\delta)n-1)} \right]
\]

If \( n > \tilde{n}_2 \), then the set of sustainable profit is
\[
\left[ 0, \frac{\delta(n-1)(1-\delta)(C-c)}{2(n(1-\delta)-1)} \right]
\]

If (9) fails, then \( \tilde{n}_1 < \tilde{n}_3 < \tilde{n}_2 \). Any profit \( \pi \in [0, \pi^M] \) can be sustained for all \( n \leq \tilde{n}_1 \).

For \( n \in (\tilde{n}_1, \tilde{n}_3] \), the set of sustainable profit is
\[
\left[ 0, \frac{(n-1)(1-\delta)((n+1)\delta+1)\delta(C-c)}{(1+\delta)(2(1-\delta)\gamma-1)(1+2\delta)n-1)} \right] \cup \left[ \frac{1-\delta}{3-2n(1-\delta)} \frac{(2n-1)(1+\delta)(U-C)-\delta n((n-1)\delta-1)(C-c)}{(1+\delta)n}, \pi^M \right]
\]

If \( n \in (\tilde{n}_3, \tilde{n}_2] \), the set of sustainable profit is
\[
\left[ 0, \frac{(n-1)(1-\delta)((n+1)\delta+1)\delta(C-c)}{(1+\delta)(2(1-\delta)\gamma-1)(1+2\delta)n-1)} \right]
\]

If \( n > \tilde{n}_2 \), then the set of sustainable profit is
\[
\left[ 0, \frac{\delta(n-1)(1-\delta)(C-c)}{2(n(1-\delta)-1)} \right].
\]

Theorem AA1 is established through construction of a number of lemmas, Lemmas AA2-AA5. From the proof of Proposition AA1, we can see that when firms charge the most effective collusive prices, a deviating firm either is forced to undercut the cartridge price (when \( p = P \)) or is indifferent between undercutting the printer price and undercutting the cartridge price (when \( p < P \)). Therefore, given that we assume that firms post the most effective collusive prices, the deviation profit is always equal to
\[
g_3(p) = \frac{(2n-1)(p-C)}{n} + \frac{\delta(C-c)}{1+\delta}.
\]
This result will be applied throughout the proof of the theorem.

Suppose for now the industry targets a per-generation industry profit of \( \pi \leq \delta(C-c) \). Applying Proposition AA1, the deviation profit is minimized at \( P = p = \bar{p} = \frac{\pi+C+\delta c}{1+\delta} \). So, for \( \pi \leq \delta(C-c) \), firms’ incentive constraint reduces to
\[
\frac{\pi}{n(1-\delta)} \geq \frac{(2n-1)}{n} \left( \frac{\pi+C+\delta c}{1+\delta} - C \right) + \frac{\delta C-c}{1+\delta}, \tag{10}
\]
which can be rewritten as
\[
2 \left( n(1-\delta) - 1 \right) \pi \leq \delta \left( n-1 \right) (1-\delta) (C-c). \tag{11}
\]
This incentive constraint is obviously satisfied if \( n \leq 1/(1-\delta) \). And for \( n > 1/(1-\delta) \), it is easier to satisfy with a lower \( \pi \). Therefore, the incentive constraint is satisfied for all \( \pi \leq \delta(C-c) \) if it is satisfied at \( \pi = \delta(C-c), \) i.e., when

\[
2(n(1-\delta)-1)\delta(C-c) \leq \delta(n-1)(1-\delta)(C-c)
\]

\[
\iff n \leq \frac{1+\delta}{1-\delta} = \hat{n}_2.
\]

And for \( n > \hat{n}_2 \), the set of sustainable profits is characterized by (11). Clearly \( \hat{n}_2 > 1/(1-\delta) \). By now we have established the following lemma:

**Lemma AA2** For all \( n \leq \hat{n}_2 \), where \( \hat{n}_2 > 1/(1-\delta) \), any profit \( \pi \in [0, \delta(C-c)] \) can be supported by tacit collusion. For all \( n > \hat{n}_2 \), any profit

\[
\pi \in [0, \frac{\delta(n-1)(1-\delta)(C-c)}{2(n(1-\delta)-1)}]
\]

(12)

can be supported by tacit collusion.

Next, suppose the industry targets a per-generation industry profit of \( \pi \in [\delta(C-c), \frac{1}{2}\frac{2n+\delta+n\delta}{n}U + \frac{1}{2}\frac{2n-\delta+n\delta}{n}C - \delta c] \). According to Proposition AA1, the most effective collusive prices are \((P,p) = (\frac{2n+\delta+n\delta}{2n\delta+n\delta}, \frac{2n\pi+(2n-\delta(n-1))C+2n\delta c}{2n\delta+n\delta})\). Therefore, tacit collusion is sustainable if and only if

\[
\frac{\pi}{n(1-\delta)} \geq \frac{(2n-1)}{n} \left( \frac{\pi + 2nC + (n+1)\delta c}{2n + (n+1)\delta} - C \right) + \frac{\delta(C-c)}{1+\delta},
\]

(13)

which can be rewritten as

\[
\left( \frac{(2n-1)(n+1)}{2n + (n+1)\delta} - \frac{1}{1-\delta} \right) \pi \leq \frac{(n-1)((n+1)\delta + 1)\delta(C-c)}{(1+\delta)(2n + (n+1)\delta)}.
\]

(14)

Since the right hand side is positive, the incentive constraint is always satisfied if:

\[
\frac{n-n\delta-n\delta^2-1-(2n+\delta+n\delta)}{2n+\delta+n\delta} < 0
\]

\[
: \quad -(n+1)(\delta + n\delta + 1) < 0
\]

\[
\Rightarrow \quad \frac{(2n-1)(n+1)}{2n + (n+1)\delta} \leq \frac{1}{1-\delta}
\]

\[
\iff n \leq \tilde{n}_0 = \frac{(1+2\delta) + \sqrt{4\delta^2 - 4\delta + 9}}{4(1-\delta)}
\]

Note that

\[
\frac{(2n-1)(n+1)}{2n + (n+1)\delta} - n = \frac{-(n+1)(\delta + n\delta + 1)}{2n + (n+1)\delta} < 0.
\]

This implies \( \tilde{n}_0 > 1/(1-\delta) \).
When $n > \hat{n}_0$, a positive profit is still sustainable but the sustainable profit is bounded from above according to (14):

$$\pi \leq \frac{(n-1)(1-\delta)(n+1)\delta(C-c)}{(1+\delta)(2(1-\delta)n^2-(1+2\delta)n-1)}. \quad (15)$$

The above incentive constraint is easier to satisfy for smaller $\pi$. Therefore, it is satisfied for all $\pi \in [\delta(C-c), \frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c]$ if it is satisfied at $\pi = \frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c$, i.e.

$$\frac{1}{2} \left( 2 + \delta + \frac{\delta}{n} \right) U + \frac{1}{2} \left( -2 + \delta - \frac{\delta}{n} \right) C - \delta c \leq \frac{(1-\delta)\delta(C-c)[n^2\delta + n - (1+\delta)]}{[1+\delta](2(1-\delta)n^2-(1+2\delta)n-1)} \quad (16)$$

which can be rewritten as

$$n \leq \frac{(\delta+1)(U-C+2\delta(U-c)) + \sqrt{(\delta+1)^2(U-C+2\delta(U-c))^2+8(1-\delta^2)(U-C+U\delta-c\delta)(U-C)}}{4(1-\delta)(U-C+U\delta-c\delta)} \equiv \tilde{n}_1.$$

For $n > \tilde{n}_1$, the sustainable profit is bounded from above by (15). The bound is at least as great as $\delta(C-c)$ if and only if $n \leq \hat{n}_2$. Summing up, we have

**Lemma AA3** For $n \leq \hat{n}_1$, where $\hat{n}_1 > 1/ (1-\delta)$, any profit $\pi \in [\delta(C-c), \frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c]$ is sustainable by tacit collusion. For $n \in (\tilde{n}_1, \tilde{n}_2)$, any profit

$$\pi \in \left[ \delta(C-c), \frac{(n-1)(1-\delta)(n+1)\delta(C-c)}{(1+\delta)(2(1-\delta)n^2-(1+2\delta)n-1)} \right]$$

is sustainable by tacit collusion.

Now suppose the firms target profit $\pi \in (\frac{1}{2} \frac{2n+\delta+n\delta}{n} U + \frac{1}{2} \frac{-2n-\delta+n\delta}{n} C - \delta c, \pi^M]$. According to Proposition 1, to support profits in this range, the most effective collusive prices are

$$(P,p) = \left( \pi + C - \delta \left( \frac{U - C + n\pi + n(2C + \delta c)}{(2 + \delta) n} - c \right), \frac{U - C + n\pi + n(2C + \delta c)}{(2 + \delta) n} \right)$$

The incentive constraint is

$$\frac{\pi}{n(1-\delta)} \geq \frac{(2n-1)}{n} \left( \frac{U - C + n\pi + n(2C + \delta c)}{(2 + \delta) n} - c \right) + \delta \frac{(C-c)}{1+\delta}$$

which can be rewritten as

$$[3 - 2n(1-\delta)] \pi \geq \frac{1 - \delta (2n-1)(1+\delta)}{1+\delta} \frac{(U - C) - \delta n ((n-1) \delta - 1)(C-c)}{n} \quad (18)$$
Note that \( n = \frac{3/2}{1 - \delta} \) satisfies the inequality if and only if \(^3\)

\[
3(2\delta - 1)\delta (C - c) \geq 4(1 - \delta^2)(U - C)
\]

\[
\iff \delta \geq \frac{3(C-c) + \sqrt{9(C-c)^2 + 16(U-C)(6(C-c) + 4(U-C))}}{4[3(C-c) + 2(U-C)]}
\]

(19)

The analysis below is divided between two cases: \( n < \frac{3/2}{1 - \delta} \) and \( n > \frac{3/2}{1 - \delta} \).

**Lemma AA4** (i) Suppose \( n < \frac{3}{2} / [2(1 - \delta)] \). If \( n \leq \bar{n}_1 \), any profit \( \pi \in (\frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, \pi^M) \) is sustainable by tacit collusion; if \( n > \bar{n}_1 \), any profit \( \pi \in \left[ \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, \pi^M \right] \) is sustainable by tacit collusion; if \( n = \bar{n}_3 \), any profit

\[
\pi \in \left[ \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, -\frac{1 - \delta}{2n(1 - \delta) - 3} \frac{(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c)}{(1 + \delta)n} \right]
\]

is sustainable.

(ii) Suppose \( n > \frac{3}{2} / [2(1 - \delta)] \). If \( n \leq \bar{n}_3 \) the any profit \( \pi \in (\frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, \pi^M) \) is sustainable; if \( n > \bar{n}_3 \), any profit

\[
\pi \in \left[ \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, \frac{1 - \delta}{2n(1 - \delta) - 3} \frac{(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c)}{(1 + \delta)n} \right]
\]

is sustainable.

**Proof** When \( n < \frac{3}{2} / [2(1 - \delta)] \), (18) provides a lower bound for \( \pi \). However, when \( n > \frac{3}{2} / [2(1 - \delta)] \), (18) provides a upper bound for \( \pi \).

When \( n < \frac{3/2}{1 - \delta} \), the incentive constraint is

\[
\pi \geq \frac{1 - \delta}{(1 + \delta)3 - 2n(1 - \delta)} \frac{(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c)}{n}
\]

(20)

The above incentive constraint is easier to satisfy for larger \( \pi \). Therefore, it is satisfied for all \( \pi \in \left( \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c, \pi^M \right) \) if it is satisfied at \( \pi = \frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c \), i.e.

\[
\frac{1}{2} \frac{2n + \delta + n\delta}{n} U + \frac{1}{2} \frac{2n - \delta + n\delta}{n} C - \delta c \geq \frac{1 - \delta}{3 - 2n + 2n\delta} \frac{(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c)}{(1 + \delta)n}
\]

(21)

\[
\delta^2 (C - c) \left( \frac{3/2}{1 - \delta} \right)^2 - [\delta (C - c) (1 + \delta) + 2 (1 + \delta) (U - C)] \left( \frac{3/2}{1 - \delta} \right) (1 + \delta) (U - C)
\]

\[
= \left( \frac{3/2}{1 - \delta} \right)^2 \frac{1}{9} [9\delta^2 (C - c) - 6 (1 + \delta) (1 - \delta) \delta (C - c) + 2 (U - C)] + 4 (1 + \delta) (U - C) (1 - \delta)^2
\]

\[
= \frac{1}{4} \left( \frac{1}{1 - \delta} \right)^2 (\delta + 2) [3 (2\delta - 1) \delta (C - c) - 4 (1 - \delta^2) (U - C)]
\]

\(^4\)When \( n = \frac{3/2}{1 - \delta} \), the incentive constraint holds if and only if (19) holds.
which can be rewritten as

\[-2 (1 - \delta) ((\delta + 1) (U - C) + \delta (C - c)) n^2 + (\delta + 1) (U - C + 2\delta (U - c)) n + (1 + \delta) (U - C) \geq 0 \tag{22}\]

This is exactly the same inequality \( n \leq \tilde{n}_3 \) as in (16). In other words, when \( n \leq \tilde{n}_1 \), the incentive constraint is satisfied for all \( \pi \in \left(\frac{1}{2} \frac{2n + n\delta}{n} U + \frac{1}{2} \frac{-2n - n\delta}{n} C - \delta c, \pi^M\right) \). When \( n > \tilde{n}_1 \), it is satisfied for

\[\pi \in \left[ \frac{1 - \delta}{1 + \delta}, \frac{1}{2} \frac{n(1 - \delta)}{2n(1 - \delta)} (2n - 1) (1 + \delta) (U - C) - \delta n ((n - 1) \delta - 1)(C - c), \pi^M\right]. \tag{23}\]

Next, suppose \( n > \frac{3}{2} \frac{1 - \delta}{1 + \delta} \). Then the incentive constraint is

\[\pi \leq \frac{1 - \delta}{3 - 2n(1 - \delta)} (2n - 1) (1 + \delta)(U - C) - \delta n ((n - 1) \delta - 1)(C - c), \tag{24}\]

This constraint is easier to satisfy if \( \pi \) is smaller. Thus, it is satisfied for all \( \pi \in \left(\frac{1}{2} \frac{2n + n\delta}{n} U + \frac{1}{2} \frac{-2n - n\delta}{n} C - \delta c, \pi^M\right) \) if it is satisfied at \( \pi = \pi^M = (U - C) + \delta (U - c) \), i.e.

\[(U - C) + \delta (U - c) \leq - \frac{1 - \delta}{2n(1 - \delta) - 3} \frac{(2n - 1) (1 + \delta)(U - C) - \delta n ((n - 1) \delta - 1)(C - c)}{(1 + \delta)n}, \]

\[n \leq \tilde{n}_3 \equiv \frac{(1 + \delta)[(5\delta + 1)(U - C) + 2\delta (C - c)] + \sqrt{(1 + \delta)^2 (5\delta + 1)(U - C) + 2\delta (C - c)]^2 + 4(1 - \delta)^2 (1 + \delta)(U - C) [2(1 + \delta)^2 (U - C) + \delta (C - c)]}}{2(1 - \delta)[2(1 + \delta)^2 (U - C) + \delta (C - c)]}. \]

When \( n > \tilde{n}_3 \), the incentive constraint is satisfied for

\[\pi \in \left[ \frac{1}{2} \frac{2n + n\delta}{n} U + \frac{1}{2} \frac{-2n - n\delta}{n} C - \delta c, \frac{1 - \delta}{2n(1 - \delta) - 3} \frac{(2n - 1) (1 + \delta)(U - C) - \delta n ((n - 1) \delta - 1)(C - c)}{(1 + \delta)n}\right]. \]

\[\text{Lemma AA5} \quad \text{If (19) holds, then } \frac{3}{2} \frac{1 - \delta}{1 + \delta} < \tilde{n}_3 \leq \tilde{n}_1. \quad \text{If (19) does not hold, then } \tilde{n}_1 < \tilde{n}_3 < \frac{3}{2} \frac{1 - \delta}{1 + \delta}. \quad \text{Also max} \{\tilde{n}_1, \tilde{n}_3\} < \tilde{n}_2. \]

**Proof.** It can be verified that \( \tilde{n}_1 \geq \frac{3}{2} \frac{1 - \delta}{1 + \delta} \) if and only if (19) holds. This can be shown by plugging \( n = \frac{3}{2} \frac{1 - \delta}{1 + \delta} \) into (22):

\[-2 (1 - \delta) ((\delta + 1) (U - C) + \delta (C - c)) \left( \frac{3/2}{1 - \delta} \right)^2 + (\delta + 1) (U - C + 2\delta (U - c)) \left( \frac{3/2}{1 - \delta} \right) + (1 + \delta) (U - C) \]

\[= \frac{1}{2 (1 - \delta)} (-4 (U - C) (1 - \delta^2) + 3\delta (C - c) (2\delta - 1)) \geq 0. \]

The last inequality holds iff (19) holds. Similarly, it can be directly verified that \( \tilde{n}_3 \geq \frac{3}{2} \frac{1 - \delta}{1 + \delta} \) if and only if (19) holds by plugging \( n = \frac{3}{2} \frac{1 - \delta}{1 + \delta} \) into (24).
From above, when \( n < \frac{3/2}{1 - \delta} \), the incentive constraint is easier to satisfy for larger \( \pi \). Since (24) supports a higher profit than (22), it holds for a wider range of \( n \). Therefore, \( \tilde{n}_3 > \tilde{n}_1 \). Similarly, when \( n > \frac{3/2}{1 - \delta} \), \( \tilde{n}_3 < \tilde{n}_1 \) holds for a similar logic.

To verify that \( \tilde{n}_2 = \frac{1 + \delta}{1 - \delta} \geq \tilde{n}_1 \), substitute \( n = \frac{1 + \delta}{1 - \delta} \) into (22)

\[
-2(1 - \delta)((\delta + 1)(U - C) + \delta(C - c))\left(\frac{1 + \delta}{1 - \delta}\right)^2 + (\delta + 1)(U - C + 2\delta(U - c))\left(\frac{1 + \delta}{1 - \delta}\right) + (1 + \delta)(U - C)
\]

\[
= \frac{1 + \delta}{1 - \delta}[-2\delta(U - C)] < 0
\]

Thus, \( \tilde{n}_2 = \frac{1 + \delta}{1 - \delta} \geq \tilde{n}_1 \). Similarly, the fact that \( \tilde{n}_2 = \frac{1 + \delta}{1 - \delta} > \tilde{n}_3 \) can be verified by substituting \( n = \frac{1 + \delta}{1 - \delta} \) into (??).

Now we are ready to prove the theorem.

**Proof of Theorem AA1:** If (19) holds, then \( \tilde{n}_3 < \tilde{n}_1 < \tilde{n}_2 \) according to Lemma A5. When \( n \leq \tilde{n}_3 < \min \{\tilde{n}_1, \tilde{n}_2\} \), the union of the profits sustainable in Lemmas AA2-AA4 is \( [0, \pi^M] \).

For \( n \in (\tilde{n}_3, \tilde{n}_1] \), the union of sets of the profits sustainable in Lemmas AA2-AA4 is

\[
\left[0, \frac{1 - \delta}{3 - 2n(1 - \delta)}(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c) \right] (1 + \delta n).
\]

The rest of the proof is completed in a similar manner.

If \( n \in (\tilde{n}_1, \tilde{n}_2] \), the union of the sets of profits sustainable is

\[
\left[0, \frac{(n - 1)(1 - \delta)(n + 1)\delta + 1)(C - c)}{(1 + \delta)(1 - \delta)n^2 - (1 + 2\delta)n - 1}\right].
\]

If \( n > \tilde{n}_2 \), the set of profits sustainable is

\[
\left[0, \frac{\delta(n - 1)(1 - \delta)(C - c)}{2n(1 - \delta) - 1}\right].
\]

If (19) fails, then \( \tilde{n}_1 < \tilde{n}_3 < \tilde{n}_2 \). Any profit \( \pi \in [0, \pi^M] \) can be sustained for all \( n \leq \tilde{n}_1 \).

For \( n \in (\tilde{n}_1, \tilde{n}_3] \), the set of sustainable profit is

\[
\left[0, \frac{(n - 1)(1 - \delta)((n + 1)\delta + 1)(C - c)}{(1 + \delta)(1 - \delta)n^2 - (1 + 2\delta)n - 1}\right] \cup \left[\frac{1 - \delta}{3 - 2n(1 - \delta)}(2n - 1)(1 + \delta)(U - C) - \delta n((n - 1)\delta - 1)(C - c) \right] (1 + \delta n), \pi^M
\]

If \( n \in (\tilde{n}_3, \tilde{n}_2] \), the set of sustainable profit is

\[
\left[0, \frac{(n - 1)(1 - \delta)((n + 1)\delta + 1)(C - c)}{(1 + \delta)(1 - \delta)n^2 - (1 + 2\delta)n - 1}\right]
\]
If $n > \bar{n}_2$, then the set of sustainable profit is

$$\left[0, \frac{\delta(n-1)(1-\delta)(C-c)}{2(n(1-\delta)-1)}\right]$$

2 Extension: Generalized Flow of Consumers

In the paper, we assume that consumers live exactly two periods and exit the market with certainty afterward. A more realistic model would allow consumers to potentially stay in the market for longer and not to exit in such an abrupt manner. In this section, we modify the main model to allow for these features. By doing so, we demonstrate that the facilitation of tacit collusion owing to firms’ aftermarket power is a general property that extends to markets wherein consumers exhibit a exit rate between zero and one. The extended model to be presented here includes, as special cases, the model analyzed in the previous sections as well as markets where the exit of established consumers exhibits a constant exit rate property. It will also be clear from my presentation in this section that the striking result that tacit collusion can be sustained among arbitrarily many firms applies as long as the market life expectancies of established customers are lower than that of new customers.

2.1 Model Modification

We first describe the entry and exit of consumers in periods starting from the second period. In every period $t \geq 2$, a measure one of consumers arrives. New consumers remain in the market in the following period with probability $\theta$. All established customers, regardless of when they arrived, remain in the market in the following period with probability $\phi \in [0, \theta]$. To ensure that the market arrives at a steady state in the second period, we assume that $\theta + \phi = 1$ and that in the first period, $1/\theta$ new customers enter the market. It is clear that starting from the second period, there are a measure one of new consumers and a measure one of established customers in every period. Note that this model captures two polar cases: (i) $\theta = \phi = 0.5$ (constant exit rate) and (ii) $\theta = 1$, $\phi = 0$ (the main model). When $\theta = \phi$, new customers and established customers have the same market life expectancy. When $\theta > \phi$, established customers have a shorter market life expectancy than new customers do.
2.2 Zero Profit Equilibrium

Following the same procedure as in Section 5.1 of the paper, we can compute the zero profit equilibrium prices. With the zero profit condition

\[(P - C) + \delta \theta \left[ 1 + \frac{\phi \delta}{1 - \delta \phi} \right] (p - c) = 0\]

and the requirement that \(P = p\), we can pin down the zero profit equilibrium prices to be

\[P^C = p^C = \frac{(1 - \delta \phi) C + \delta \theta c}{1 - \delta \phi + \delta \theta}. \quad (25)\]

2.3 Tacit Collusion

By serving both the foremarket and aftermarket, a monopolist could earn from each generation of consumers

\[\pi^M = U - C + \delta \theta \frac{(U - c)}{1 - \delta \phi}.\]

We focus the remainder of this section on deriving the maximum steady-state profit firms can sustain through tacit collusion. The tacit collusion we consider is supported by trigger strategies in which firms revert to the zero profit equilibrium pricing (25) as soon as any firm deviates. We prove the following result:

**Proposition AA2** (i) For all \((\theta, \phi) \in [0, 1]^2\) such that \(\phi = 1 - \theta\) and \(\phi \leq \theta\), there exists \(\bar{n} > \frac{1}{1 - \delta}\) such that any per-generation industry profit \(\pi \in [0, \pi^M]\) is sustainable if and only if \(n \leq \bar{n}\). Moreover, (ii) for all \(n \geq 2\), any per-generation industry profit

\[\pi \in \left[0, \min\{\pi^M, \frac{\delta (\theta - \phi) (C - c)}{2(1 - \delta \phi)}\}\right]\]

can be supported by tacit collusion.

**Proof.** First, we prove that any profit ranging from zero to the monopoly profit can be supported among a larger number of firms than \(1/(1 - \delta)\). For this purpose, we do not have to fully characterize the conditions under which profitable tacit collusion is sustainable. Instead, we will just derive *sufficient conditions* for sustainability of tacit collusion.

Given a print-cartridge price pair \((P, p)\), where \(p \leq P\), the equilibrium per-generation industry profit will be

\[\pi = (P - C) + \delta \theta \frac{p - c}{1 - \delta \phi}.\]
which can be rewritten as

\[ P = C + \pi - \delta \theta \frac{p - c}{1 - \delta \phi}. \]  

(26)

If firms charge the same price for the printer and cartridge, then

\[ P = p = \bar{p} \equiv \frac{(1 - \delta \phi)(\pi + C) + \delta \theta c}{1 + \delta (\theta - \phi)}. \]  

(27)

Note that, however, the steady state profit each firm earns \textit{per-period} will be \((P - C + p - c)/n\) instead of \(\pi/n\). This is because in every period starting from the second period, each firm will serve a measure \(1/n\) of established customers and a measure \(1/n\) of new customers. Therefore, the discounted profit of each firm, inclusive of profit from its established customers is

\[ \frac{P - C + p - c}{n (1 - \delta)} = \frac{C + \pi - \delta \theta \frac{p - c}{1 - \delta \phi} - C + p - c}{n (1 - \delta)} = \frac{\pi + \frac{1 - \delta \theta - \delta \phi}{1 - \delta \phi}(p - c)}{n (1 - \delta)}. \]  

(28)

If a firm deviates by setting the printer price arbitrarily close to but less than \(\min\{P, U\}\), then it will capture a measure one of new consumers, earning from them an immediate profit of \((\min\{P, U\} - C)/n\) and, in all future periods, a discounted profit of \(\delta \theta \left(\frac{p - c}{1 - \delta \phi}\right)\), based on the expectation that a price war will begin in the following period. When the firm deviates, it will also raise its cartridge price to arbitrarily close to \(\min\{U, P\}\) and earn from its \(1/n\) established customers an immediate profit arbitrarily close to \((\min\{U, P\} - c)/n\) [instead of \(p/c\) as in equilibrium] and a discounted future profit of \(\frac{\delta \phi}{n} \left(\frac{p - c}{1 - \delta \phi}\right)\). This gives rise to a deviation profit of

\[ (\min\{P, U\} - C) + \frac{(\min\{P, U\} - c)}{n} + \delta \left(\theta + \frac{\phi}{n}\right) \left(\frac{p - c}{1 - \delta \phi}\right) \]

\[ \leq (P - C) + \frac{(P - c)}{n} + \delta \left(\theta + \frac{\phi}{n}\right) \left(\frac{p - c}{1 - \delta \phi}\right) \]

\[ = \frac{(n + 1)P - nC - c}{n} + \delta \left(\theta + \frac{\phi}{n}\right) \left(\frac{C - c}{1 - \delta \phi + \delta \theta}\right) \]

\[ = \frac{(n + 1)}{n} \left(C + \pi - \delta \theta \frac{p - c}{1 - \delta \phi}\right) - C - \frac{c}{n} + \delta \left(\theta + \frac{\phi}{n}\right) \left(\frac{C - c}{1 - \delta \phi + \delta \theta}\right) \equiv g_1(p, \pi). \]

The inequality is trivial, the first equality follows (25), and the second equality follows (26).

If the deviating firm instead sets the printer price arbitrarily close to but less than \(p\), then it will earn a profit of \((p - C)\) from the new consumers and a profit of \(\frac{(p - c)}{n}\) from its own established customers. It will also steal a measure \((1 - \frac{1}{n})\) of established customers from its competitors, earning from them a profit of \((1 - \frac{1}{n})(p - C)\). The deviating firm will also earn a discounted profit of \(\delta \theta \left(\frac{p - c}{1 - \delta \phi}\right) = \frac{\delta \theta (C - c)}{1 - \delta \phi + \delta \theta}\) from
the new customers and a discounted profit of \( \delta \phi \left( \frac{p_c - c}{1 - \delta \phi} \right) = \frac{\delta \phi (C - c)}{1 - \delta \phi + \delta \theta} \) from the established customers. Therefore, such deviation leads to a profit of

\[
g_2(p) \equiv \left( 2 - \frac{1}{n} \right) (p - C) + \frac{p - c}{n} + \delta \frac{(\theta + \phi) (C - c)}{1 - \delta \phi + \delta \theta}.
\]

Therefore, for \( p < P \) (i.e., \( p < \tilde{p} \)), the deviation profit is no larger than \( \max \{ g_1(p, \pi), g_2(p) \} \). Suppose \( P = p \), then the deviation profit is necessarily \( g_2(p) \). Summing up, the deviation profit does not exceed

\[
\hat{D}(p, \pi) = \begin{cases} 
\max \{ g_1(p, \pi), g_2(p) \} & \text{if } p < \tilde{p}, \\
g_2(p) & \text{if } p = \tilde{p}.
\end{cases}
\]

Let \( p = \tilde{p}_{12} \) solve \( g_1(p, \pi) = g_2(p) \). It can be verified that

\[
\tilde{p}_{12} = \frac{1 - \delta \phi}{2n(1 - \delta \phi + \delta \theta(n + 1))} \left( (n + 1) \pi + 2nC + \frac{(n + 1) \delta \theta c}{1 - \delta \phi} - \delta \phi (n - 1) \frac{C - c}{1 + \delta (\theta - \phi)} \right). \tag{29}
\]

Also, we know that \( g_1 \) is decreasing in \( p \) and \( g_2 \) is increasing in \( p \) and thus

\[
\max \{ g_1(p, \pi), g_2(p) \} = \begin{cases} 
g_1(p, \pi) & \text{if } p \leq \tilde{p}_{12}, \\
g_2(p) & \text{if } p > \tilde{p}_{12},
\end{cases}
\]

and \( \max \{ g_1(p, \pi), g_2(p) \} \) is minimized at \( p = \tilde{p}_{12} \).

By comparing (27) with (29), it can be verified that

\[
\tilde{p}_{12} < \tilde{p} \text{ if and only if } \pi > \frac{\delta (\theta - \phi) (C - c)}{1 - \delta \phi}. \tag{30}
\]

First suppose the industry targets some \( \pi \geq \frac{\delta (\theta - \phi) (C - c)}{1 - \delta \phi} \) so that \( \tilde{p}_{12} \leq \tilde{p} \). In this case,

\[
\hat{D}(p, \pi) = \begin{cases} 
g_1(p, \pi) & \text{if } p < \tilde{p}_{12}, \\
g_2(p) & \text{if } p \in [\tilde{p}_{12}, \tilde{p}].
\end{cases}
\]

and \( \hat{D}(p, \pi) \) is minimized at \( p = \tilde{p}_{12} \). Suppose that firms support this profit by setting \( p = \tilde{p}_{12} \) and setting \( P \) according to (26). In this case, the sufficient condition for sustainability of tacit collusion is

\[
\frac{\pi + \frac{1 - \delta \theta - \delta \phi}{1 - \delta \phi} (\tilde{p}_{12} - c)}{n(1 - \delta)} \geq \hat{D}(\tilde{p}_{12}, \pi).
\]

By plugging (29) into \( \hat{D}(\tilde{p}_{12}, \pi) \), we can show that

\[
\frac{\pi + \frac{1 - \delta \theta - \delta \phi}{1 - \delta \phi} (\tilde{p} - c)}{n(1 - \delta)} - \hat{D}(\tilde{p}_{12}, \pi) = \frac{(1 - \delta \phi) K_1 \pi}{(1 - \delta) n (n (2 - 2 \delta \phi + \delta \theta) + \delta \theta)} + \frac{\delta K_2 (C - c)}{17 (1 - \delta) n (1 + \delta (\theta - \phi)) (n (2 - 2 \delta \phi + \delta \theta) + \delta \theta)}
\]

By plugging (29) into \( \hat{D}(\tilde{p}_{12}, \pi) \), we can show that

\[
\frac{\pi + \frac{1 - \delta \theta - \delta \phi}{1 - \delta \phi} (\tilde{p} - c)}{n(1 - \delta)} - \hat{D}(\tilde{p}_{12}, \pi) = \frac{(1 - \delta \phi) K_1 \pi}{(1 - \delta) n (n (2 - 2 \delta \phi + \delta \theta) + \delta \theta)} + \frac{\delta K_2 (C - c)}{17 (1 - \delta) n (1 + \delta (\theta - \phi)) (n (2 - 2 \delta \phi + \delta \theta) + \delta \theta)}
\]
where
\[
K_1 = (1 + 2\delta) n + 1 - 2(1 - \delta) n^2 \\
K_2 = (1 - \delta)(\theta - \phi) \delta n^2 - (\theta + 3\phi - \theta\delta + 2\delta\phi + 2\theta^2\delta - 3\delta\phi^2 - \theta\delta\phi - 2) n \\
- (\theta - \phi - \theta^2\delta^2 - \theta\delta + \theta^2\delta + \delta\phi^2 + \theta\delta^2\phi)
\]

By plugging \(\theta = 1 - \phi\) into \(K_2\) and recalling that \(\phi \leq 0.5\), we have
\[
K_2 = (1 - \delta)(1 - 2\phi)(n - 1)((n + 1)\delta(1 - \phi) + 1) \geq 0.
\]

Moreover, it can be verified that
\[
K_1 \geq 0 \\
\Leftrightarrow n \leq \tilde{n}_1 = \frac{\sqrt{4\delta^2 - 4\delta + 9 + 2\delta + 1}}{4(1 - \delta)},
\]

In other words, if \(\theta \geq \phi\), then any profit in the range \([\frac{\delta(\theta-\phi)(C-c)}{1-\delta\phi}, \pi^M]\) is sustainable for all \(n \leq \tilde{n}_1\), and \(\tilde{n}_1 > \frac{1}{1-\delta}\) because
\[
\frac{\tilde{n}_1 - \frac{1}{1 - \delta}}{4(1 - \delta)} = \frac{\sqrt{4\delta^2 - 4\delta + 9 + 2\delta + 1} - 4}{4(1 - \delta)} = \frac{\sqrt{4\delta^2 - 4\delta + 9 - (3 - 2\delta)} - \sqrt{4\delta^2 - 4\delta + 9 - 12\delta + 9}}{4(1 - \delta)} > 0.
\]

In the case that \(\theta = \phi\), the lower bound of \([\frac{\delta(\theta-\phi)(C-c)}{1-\delta\phi}, \pi^M]\) is 0. Then the proof of part (i) of the proposition can be completed by choosing \(\tilde{n} = \tilde{n}_1\).

Now, consider the case that \(\theta > 0.5 > \phi\) so that \(\frac{\delta(\theta-\phi)(C-c)}{1-\delta\phi} > 0\). Suppose the industry targets some \(\pi < \frac{\delta(\theta-\phi)(C-c)}{1-\delta\phi}\). In this case \(\bar{p} < \tilde{p}_{12}\) and
\[
\hat{D}(p, \pi) = \begin{cases} 
g_1(p, \pi) & \text{if } p < \bar{p}, \\
g_2(p) & \text{if } p = \bar{p},
\end{cases}
\]

where \(g_2(\bar{p}) < g_1(\bar{p}, \pi)\) because \(\bar{p} < \tilde{p}_{12}\). Since \(g_1(p, \pi)\) is decreasing, \(\hat{D}(p, \pi)\) is minimized at \(p = \bar{p}\).

Suppose the industry sets \(P = p = \bar{p}\) to support the targeted \(\pi\). In this case, the sufficient condition for sustainability of tacit collusion is
\[
\frac{2\bar{p} - (C + c)}{n(1 - \delta)} \geq \left(2 - \frac{1}{n}\right)(\bar{p} - C) + \frac{(\bar{p} - c)}{n} + \delta \frac{(\theta + \phi)(C - c)}{1 - \delta\phi + \delta\theta}.
\]
By plugging (27) and \( \theta = 1 - \phi \) into (31), the latter can be simplified as

\[
n \left( \frac{\pi - \delta (1 - 2\phi) (C - c)}{2(1 - \delta \phi)} \right) \leq \frac{1}{(1 - \delta)} \left( \frac{\pi - \delta (1 - \delta) (1 - 2\phi) (C - c)}{2(1 - \delta \phi)} \right).
\]

For \( \pi \in \left( \frac{\delta(1 - \phi)(C - c)}{2(1 - \delta \phi)}, \frac{\delta(1 - \phi)(C - c)}{1 - \delta \phi} \right) \), (32) can be further rewritten as

\[
n \leq \hat{n}_2 \equiv \frac{1}{(1 - \delta)} \frac{2(1 - \delta \phi) \pi - \delta (1 - \delta) (1 - 2\phi) (C - c)}{2(1 - \delta \phi) \pi - \delta (1 - 2\phi) (C - c)}.
\]

It can be verified that \( \phi < 0.5 \) implies

\[
\frac{2(1 - \delta \phi) \pi - \delta (1 - \delta) (1 - 2\phi) (C - c)}{2(1 - \delta \phi) \pi - \delta (1 - 2\phi) (C - c)} > 1.
\]

Therefore,

\[
\hat{n}_2 > \frac{1}{1 - \delta}.
\]

For \( \pi \in \left( \frac{\delta(1 - \phi)(1 - 2\phi)(C - c)}{2(1 - \delta \phi)}, \frac{\delta(1 - \phi)(1 - 2\phi)(C - c)}{2(1 - \delta \phi)} \right) \), (32) holds for all \( n \) because the LHS of the equation is negative while its RHS is positive for all \( n \). For \( \pi \in \left[ 0, \frac{\delta(1 - \phi)(1 - 2\phi)(C - c)}{2(1 - \delta \phi)} \right] \), (32) becomes

\[
n \geq \frac{1}{(1 - \delta)} \frac{\delta (1 - \delta) (1 - 2\phi) (C - c) - 2 (1 - \delta \phi) \pi}{\delta (1 - 2\phi) (C - c) - 2 (1 - \delta \phi) \pi} = \frac{\delta (1 - 2\phi) (C - c) - 2 (1 - \delta \phi) \pi}{\delta (1 - 2\phi) (C - c) - 2 (1 - \delta \phi) \pi} (\leq 1),
\]

which is always satisfied. So, in the case that \( \theta > 0.5 > \phi \), any profit \( \pi \in [0, \pi^M] \) can be supported by tacit collusion for all \( n \leq \min\{\hat{n}_1, \hat{n}_2\} \), where \( \min\{\hat{n}_1, \hat{n}_2\} > 1/(1 - \delta) \). Therefore, we can complete the proof of part (i) of the proposition by setting \( \bar{n} = \min\{\hat{n}_1, \hat{n}_2\} \).

Note that (32) can be rewritten as

\[
2 (1 - \delta \phi) ((1 - \delta) n - 1) \pi \leq \delta (1 - \delta) (n - 1) (1 - 2\phi) (C - c).
\]

This holds for all \( n \leq 1/(1 - \delta) \) and when \( n > 1/(1 - \delta) \), it hold if and only if

\[
\pi \leq \frac{\delta (1 - \delta) (n - 1) (1 - 2\phi) (C - c)}{2(1 - \delta \phi) ((1 - \delta) n - 1)} = \frac{\delta (1 - \delta) (n - 1) (\theta - \phi) (C - c)}{2(1 - \delta \phi) ((1 - \delta) n - 1)}.
\]

\[
\frac{d}{dn} \left( \frac{\delta (1 - \delta) (n - 1) (\theta - \phi) (C - c)}{2(1 - \delta \phi) ((1 - \delta) n - 1)} \right) = -\frac{\delta^2 (1 - \delta) (\theta - \phi) (C - c)}{2(1 - \delta \phi) (n (1 - \delta) - 1)^2} < 0
\]

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As $n$ approaches infinity, the upper bound on $\pi$ becomes

$$\lim_{n \to \infty} \frac{\delta (1 - \delta) (n - 1) (\theta - \phi) (C - c)}{2 (1 - \delta \phi) ((1 - \delta) n - 1)} \leq \frac{\delta (\theta - \phi) (C - c)}{2 (1 - \delta \phi)}.$$

Therefore, for all $n \geq 2$, any $\pi \in \left[0, \frac{\delta (\theta - \phi) (C - c)}{2 (1 - \delta \phi)}\right]$ can be supported by tacit collusion. Obviously, $\pi$ cannot exceed $\pi^M$ as well. This completes the proof of part (ii) of the proposition.

Proposition AA2 shows that constrained aftermarket power allows a larger number of firms to sustain any profit between zero and the monopoly profit, whether established customers have a shorter market life expectancy than new consumers or not. This is because regardless of the rates at which customers exit the market, it remains true that after a deviating firm steals the new generation of consumers from its competitors, it will not be able to charge these customers the equilibrium cartridge price in the following period. In other words, aftermarket power still prevents a deviating firm from stealing the entire industry profit from one generation of customers.

However, since $\frac{\delta (\theta - \phi) (C - c)}{2 (1 - \delta \phi)} > 0$ if and only if $\theta > \phi$, the proposition above also clarifies the fact that the sustainability of profitable tacit collusion among arbitrarily many firms of any discount factor relies on the property that established customers exit at a higher hazard rate than new customers, i.e., established customers have a shorter market life expectancy. This property can be a consequence of consumers’ market lifetimes being finite or due to the fact that the products are targeted to consumers of a particular age group (e.g., entry level printers targeted to college students).

3 Commitment to Future Cartridge Price: Case of Uncertain Demand and Unverifiable Quality of Cartridge

For most parts of the paper, we assumed firms cannot commit to the cartridge price at all. In Section 7.2 of the paper, we considered the other polar extreme in which firms can efficiently make commitment to the cartridge price (e.g., through the use of long-term contracts). In reality, there may be practical considerations which cause such commitment to be inefficient. For example, a natural way to commit to the cartridge price is to sell the printer and the replacement cartridge as a bundle. If consumers’ demands for the replacement cartridge is uncertain, however, then this commitment will be inefficient. Similarly, if the quality of cartridge is not fully verifiable, then consumers may not trust that a firm that sells the cartridge through a long-term contract will produce the cartridge at high quality at the
time of delivery. And uncertainty of production cost of cartridge may also render long-term contract inefficient. We believe these and other practical considerations that limit firms’ ability to commit to the aftermarket price efficiently are relevant, so it is important to analyze the case of inefficient commitment.

In this extension, we consider a model with the added elements that a consumer’s demand for cartridge in the second period is uncertain and that the quality of cartridge is observable but unverifiable. With these assumptions, commitment to the cartridge price cannot be efficiently done. We formally prove that in this setting, there exist equilibria in which the industry earns supranormal profit for arbitrarily large \( n \). We do not fully characterize the SPNE payoff set because it is not the focus of this extension.

**Model Modification** The quality of the cartridge is determined by the production function

\[
q(k)
\]

where \( k \geq 0 \) is the production cost incurred by the firm. Assume \( q(0) = 0 \) and \( q(c) = U \). Also suppose \( q'(k) > 1 \) for \( k < c \), \( q'(c) = 1 \), and \( q'(k) < 1 \) for \( k > c \). Therefore, it is socially optimal to produce at the quality \( q = U \) at the cost \( c \). Now we specify the key modification to the main model. Consumers’ valuations in the second period of their market life are random. With probability \( \alpha \in (0,1) \), an established customer values the compatible cartridge at \( q \) and a brand new printer at \( U \); and with probability \( (1 - \alpha) \), the established customer values both the cartridge and a new printer at zero. The cartridge’s and printer’s values to a consumer in the second period is unknown to everyone in the first period of the consumer’s market life. The consumer learns the values to him between the first period and the second period of his market life. Quality of a cartridge is *observable prior to consumption but non-verifiable*. We retain the assumption that consumers are sophisticated.

**Zero Profit Equilibrium without Price Commitment** Consider an equilibrium in which firms do not commit to the cartridge price on the equilibrium path when they sell the printer so consumers purchase the printer and cartridge separately in the first and second period of their life (on the equilibrium path). Define \( \tilde{p}^C \) by

\[
\tilde{p}^C - C + \delta \alpha (\tilde{p}^C - c) = 0, \\
\tilde{p}^C = \frac{C + \delta \alpha c}{1 + \delta \alpha}.
\]
It can be checked that it is an equilibrium that firms set \( P = p = \tilde{p}^C \). At these prices, old consumers buy a cartridge with probability \( \alpha \), firms make zero profit over a consumer’s lifetime and no firm can profitably deviate.

**Collusive Equilibrium** Next we consider a collusive outcome in which following any deviation, firms revert to the zero profit equilibrium. While firms are free to offer any contract with or without bundling, we will construct an equilibrium in which firms do not offer bundles on the equilibrium path, and derive conditions under which this equilibrium is robust to any deviation, including deviations using bundled offers.

Given that the quality of a cartridge is observable once produced, if a consumer observes the quality before purchasing, then she is willing to pay up to her valuation which is \( q(k) \) with probability \( \alpha \) and 0 with probability \( 1 - \alpha \). However, since the quality is non-verifiable, if a consumer pre-orders a cartridge through any contract which may require a full or partial advance payment from the consumer, the firm can always ignore the contract and produce a cartridge of zero quality \( q(0) = 0 \) to meet the contract requirement. Then, to earn profit from its established customers’ second-period demand for a cartridge, it can at the same time offer another cartridge at the efficient quality \( q = U \), calling it a “premium” version of the product. Since its customers’ outside option is to purchase from another firm a printer of quality \( U \) at the price \( \tilde{p}^C \), in producing the premium version, it will choose \( k \) such that

\[
q(k) - \tilde{p} = U - \tilde{p}^C.
\]

The corresponding profit from these consumers is

\[
\tilde{p} - k = q(k) - k - (U - \tilde{p}^C)
\]

which is maximized at \( k = c \) and \( q = U \). This is exactly the same quality it will choose if it does not make any commitment in price or quality at the moment of deviation. Therefore, it is without loss of generality to assume that the deviating firm either deviate by offering just a printer or by offering a bundle with a printer and a cartridge at the quality \( U \).

**Proposition AA3** *Any profit per generation of consumers*

\[
\pi \in \left(0, \min\{(1 - \delta^2 \alpha^2) c, \delta \alpha^2 \frac{C - c}{1 + \alpha}\}\right)
\]

is sustainable among any number of firms.
Proof Suppose firms set $P = p \in [\hat{p}^C, C]$ on the equilibrium path so that they earn positive profits. When consumers observe a deviation, they expect all firms to charge $\hat{p}^C$ for both the printer and the cartridge in the following period. Our assumption that the cartridge quality is unverifiable implies that firms cannot profit from deviating by offering new consumers a pre-order of cartridge at the time of deviation more than deviating by offering just a printer. Consumers rationally infer that the deviating firm making such an offer can deliver a zero-quality cartridge to satisfy the pre-order contract and separately sell to them a cartridge at quality $U$ and price $\hat{p}^C$ in the following period. This way, they will end up paying $\hat{p}^C$ in the following period again.

Now suppose a firm deviates by offering a bundle containing a new printer and a cartridge at quality $U$, priced at $P'_B$. The deviating firm attracts new customers if and only if

$$U - P + \delta \alpha (U - \hat{p}^C) \leq U + \delta \alpha U - P'_B, \text{ for } P'_B > p$$

$$P'_B \leq P + \delta \alpha \hat{p}^C.$$ 

This is because consumers have to pay $P$ to a nondeviating firm and, if they turn out to place a positive value on the high-quality cartridge, which happens with probability $\alpha$, they will be paying $\hat{p}^C$ in the following period. As long as the bundled price is higher than the equilibrium cartridge price $p (= P)$, it will not attract competitors’ established customers. Since the bundle costs the deviating firm $C + c$, this part of the deviation profit is $P'_B - (C + c)$. Apart from this offer, the deviating firm continues to charge its established customers who have a positive valuation for the high-quality cartridge the equilibrium cartridge price $p = P$ as it would do on the equilibrium path. For a deviation to be unprofitable, it requires that

$$\frac{\alpha P - c}{n} + \frac{P'_B - (C + c)}{n} \leq \alpha \frac{P - c}{n} + \frac{P - C + \delta \alpha (P - c)}{n (1 - \delta)}. $$

This condition is satisfied for all $n$ if and only if

$$P'_B \leq C + c.$$
In other words, if
\[
P + \delta \alpha \frac{C + \delta \alpha c}{1 + \delta \alpha} \leq C + c
\]
\[
P \leq C + c - \delta \alpha \frac{C + \delta \alpha c}{1 + \delta \alpha}
\]
then, for all \(n\), there does not exist a profitable deviation with a bundled offer.

If a firm deviates by offering a printer instead of a bundle, then the printer offer will attract all consumers, including competitors’ established customers, if and only if \(P' \leq P\). The firm can, however, prevents its own established customers from buying the new printer by offering the cartridge at a price infinitesimally lower than \(P'\). The deviation profit will be
\[
\alpha \left( \frac{P - c}{n} + (n - 1) \frac{P - C}{n} \right) + P - C + \delta \alpha \left( P' - c \right),
\]
where \(\alpha (P - c)/n\) comes from its established customer, \(\alpha (n - 1) (P - C)/n\) comes from competitors’ established customers and the rest comes from new consumers. Deviation with a printer offer is unprofitable if and only if
\[
\alpha \left( \frac{P - c}{n} + (n - 1) \frac{P - C}{n} \right) + P - C + \delta \alpha \left( P' - c \right) \leq \frac{P - c}{n} + \frac{P - C + \delta \alpha (P - c)}{n(1 - \delta)}.
\]
This holds for all \(n\) if and only if
\[
(1 + \alpha) (P - C) + \delta \alpha \left( \frac{C + \delta \alpha c}{1 + \delta \alpha} - c \right) \leq 0
\]
\[
P \leq C - \frac{\delta \alpha}{1 + \alpha} \left( \frac{C + \delta \alpha c}{1 + \delta \alpha} - c \right).
\]
Therefore, tacit collusion is sustainable among any number of firms if and only if
\[
P \leq \min \left\{ C - \delta \alpha \frac{C + \delta \alpha c}{1 + \delta \alpha}, C - \frac{\delta \alpha}{1 + \alpha} \left( \frac{C + \delta \alpha c}{1 + \delta \alpha} - c \right) \right\}
\]
At \(P = C + c - \delta \alpha \frac{C + \delta \alpha c}{1 + \delta \alpha}\), the industry’s profit from each generation of consumers is
\[
(1 + \delta \alpha) \left[ C + c - \delta \alpha \frac{C + \delta \alpha c}{1 + \delta \alpha} \right] - C - \delta \alpha c
\]
\[
= (1 - \delta^2 \alpha^2) c.
\]
At \(P = C - \frac{\delta \alpha}{1 + \alpha} \left( \frac{C + \delta \alpha c}{1 + \delta \alpha} - c \right)\), the industry’s profit from each generation of consumers is
\[
(1 + \delta \alpha) \left[ C - \frac{\delta \alpha}{1 + \alpha} \left( \frac{C + \delta \alpha c}{1 + \delta \alpha} - c \right) \right] - C - \delta \alpha c
\]
\[
= \frac{\delta \alpha^2 c}{1 + \alpha}.
\]
Therefore, any profit per generation of consumers

\[ \pi \in \left(0, \min\{ (1 - \delta^2 \alpha^2) c, \delta \alpha^2 \frac{C - c}{1 + \alpha} \} \right) \]

is sustainable among any number of firms. ■